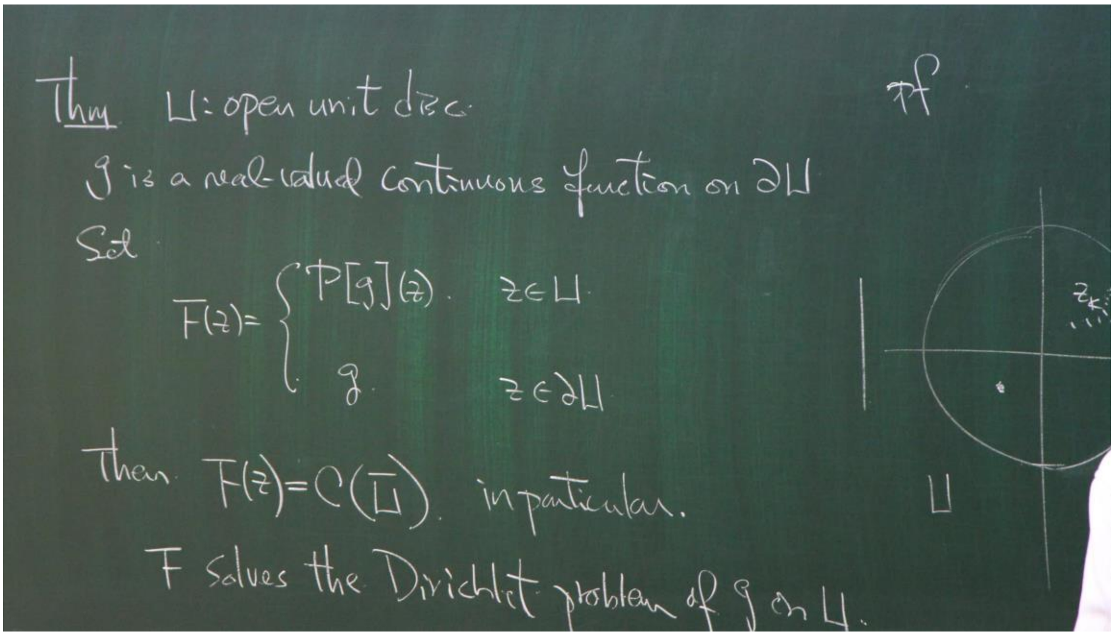
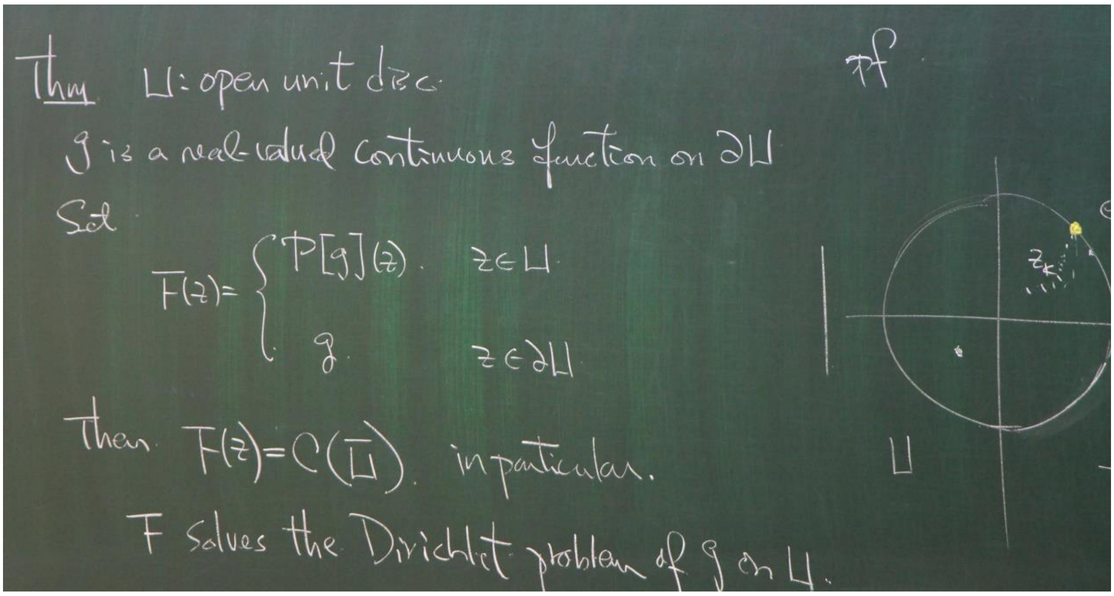
【10920 程守慶教授複變數函數論 / 第 16 堂版書】





Given $\varepsilon > 0$.

Pf

$$F(z_k) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\, g(e^{it})\,dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\, g(e^{it_0})\,dt.$$

$z_k = r_k e^{i\theta_k}$

$r_k < 1$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\left(g(e^{it}) - g(e^{it_0})\right)dt$$

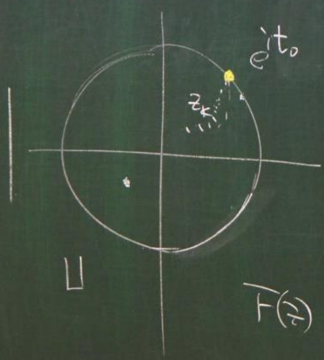
...s function on ∂U

$\in U$

$\in \partial U$

particular.

...t problem of g on U.



Given $\varepsilon > 0$.

$$F(z_k) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\, g(e^{it})\,dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\, g(e^{it_0})\,dt.$$

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Given $\varepsilon > 0$.

e^{it_0}

$z_k = r_k e^{i\theta_k}$

$r_k < 1$

$F(z)$

$$F(z_k) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\, g(e^{it})\,dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\, g(e^{it_0})\,dt.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\left(g(e^{it}) - g(e^{it_0})\right)dt$$

Given $\varepsilon > 0$. $\exists\, 2\delta > 0$. s.t.

$$|g(e^{it}) - g(e^{it_0})| < \varepsilon \quad \text{if } |e^{it} - e^{it_0}| < 2\delta.$$

$$|\mathrm{I}| \le \frac{1}{2\pi}\int_{|e^{it}-e^{it_0}|<2\delta} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\, |g(e^{it}) - g(e^{it_0})|\,dt.$$

$$< \frac{\varepsilon}{2\pi}\int_{|e^{it}-e^{it_0}|<2\delta} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\,dt$$

Given $\varepsilon > 0$.

pf.

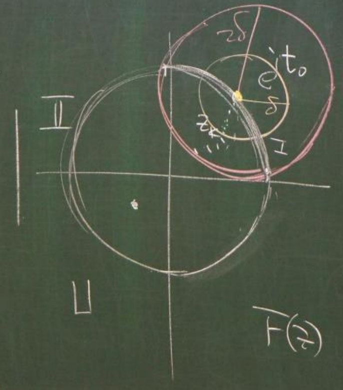
...nuous function on ∂U

$z \in U$

$z \in \partial U$

in particular.

...chlet problem of g on U.



$F(z)$

$$F(z_k) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int \cdots$$

$z_k = r_k e^{i\theta_k}$

$r_k < 1$

$= \cdots$

$= \cdots$

$= \cdots$

Given $\varepsilon > 0$. $\exists\ 2\delta > 0$ s.t.

$$|g(e^{it}) - g(e^{it_0})| < \varepsilon \quad \text{if } |e^{it} - e^{it_0}| < 2\delta.$$

$$|\mathrm{I}| \le \frac{1}{2\pi}\int_{|e^{it}-e^{it_0}|<2\delta} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\,|g(e^{it}) - g(e^{it_0})|\,dt.$$

$$< \frac{\varepsilon}{2\pi}\int_{|e^{it}-e^{it_0}|<2\delta} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\,dt \le \frac{\varepsilon}{2\pi}\underbrace{\int_0^{2\pi} \frac{1-|z_k|^2}{|e^{it} - r_k e^{i\theta_k}|^2}\,dt}_{=\ \varepsilon}$$

ous function on ∂U

$z \in U$

$z \in \partial U$

n particular.

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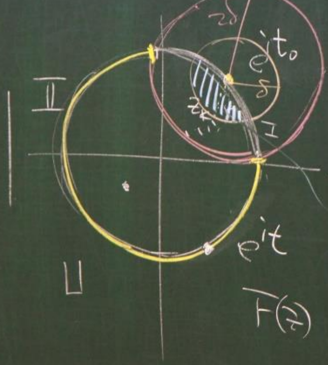
Pf.



$$|z_k - e^{it_0}| < \delta$$

Given $\varepsilon > 0$.

$$F(z_k) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi}$$

$z_k = r_k e^{i\theta_k}$

$r_k < 1$

$$= \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}($$

$$= I + II$$

$$|II| \le \frac{1}{2\pi}\int_{|e^{it}-e^{it_0}|\ge 2\delta} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\left|g(e^{it})-g(e^{it_0})\right| dt$$

$$\le \frac{2M}{2\pi}\int_{|e^{it}-e^{it_0}|\ge 2\delta} \frac{1-|z_k|^2}{|e^{it}-r_k e^{i\theta_k}|^2}\, dt$$

$$\le \frac{M}{\pi\delta^2}\int_{|e^{it}-e^{it_0}|\ge 2\delta} (1-|z_k|^2)\, dt < \varepsilon.$$

if z_k is suff. close to e^{it_0}.

$|g(e^{it})| \le M \quad \forall t \in \mathbb{R}$

$z_k \xrightarrow{\in U} e^{it_0}$ Means $r_k = |z_k| \to 1$

Assume $|z_k - e^{it_0}| < \delta$

∂U

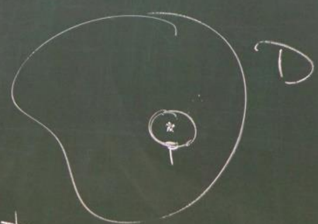
Thm. $D \subseteq \mathbb{C}$ domain.

f is a continuous function on D.

(real-valued).

Suppose f satisfies mean value property.

$\Rightarrow$ f is harmonic on D.



D

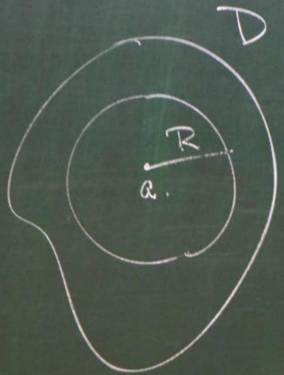
u = harmonic

u : real

$$\mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,u(re^{it})\,dt.\right)$$

$$P\left[u\big|_{|z-a|=r}\right](z)-u(z)\equiv 0.$$

pf.



$\overline{B}(a;R)\subseteq D$.

Consider

$$P\left[u\big|_{|z-a|=R}\right](z)=h(z).$$

$h=u$ on $|z-a|=R$

show $h=u$ on $B(a;R)$

set $g = h - u$. on $B(a;R)$

① $g(z)=0$. on $z \in \partial B(a;R)$

$\therefore$ let $m = \max\limits_{z\in \overline{B}(a;R)} g(z)$

If $m>0$. Set $E=\{z\in \overline{B}(a;R) \mid g(z)=m\}$ Compact set in $\overline{B}(a;R)$.

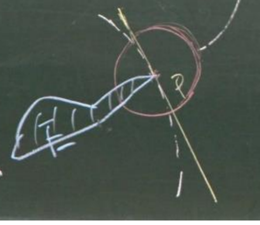
$E \cap \partial B(a;R) = \phi$.

let $p \in E$. s.t. $|p-a| \geq |z-a|$. $\forall z \in E$.

ic

u: real

$\mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}u(re^{it})\,dt\right)$.



pf.

$\exists\ r_n>0,\ r_n \to 0$

$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p+r_n e^{i\theta})\,d\theta$

$\overline{B}(a;R) \subseteq$

Consider

$P[u|$

$h=u$ on

show $h=u$ on

$B(a;R)$

$\in \partial B(a;R)$

$$m = g(p) = \frac{1}{2\pi}\int_0^{2\pi} (h-u)(p+r_n e^{i\theta})\, d\theta < m.$$

$g(z)$

$)$

$= \{ z \in \overline{B}(a;R) \mid g(z) = m \}$ Compact set in $\overline{B}(a;R)$.

$, R) = \phi.$

$a| \geq |z-a|.\ \forall z \in E.$

$\therefore\ \underline{m \leq 0}.$ $\quad h \leq u$. on $\overline{B}(a;R)$.

$) \subseteq B(a;R)$

$u-h$ $\quad h \geq u$. " $\quad \Rightarrow h \equiv u$.

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e property.

Thm let D be a domain.

$p \in D$.

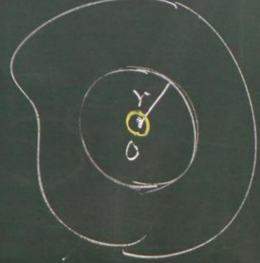
Suppose u is a real-valued harmonic function on $D \setminus \{p\}$.

and u is <u>bounded</u> on $D \setminus \{p\}$

Then u can be redefined at p, s.t. u becomes harmonic on D

pf Assume $p=0$.

$\overline{B}(0;r) \subseteq D$.



Consider

$$P[u|_{|z|=r}](z) = h(z).$$

$h(z) = u(z)$, $|z| = r$

let $\varphi(z) = h(z) - u(z)$ or $\varphi(z) = u(z) - h(z)$

For any $\varepsilon > 0$. Consider

$$\varphi_\varepsilon(z) = \varphi(z) + \varepsilon \ln \frac{|z|}{r}.$$

$z) - h(z)$

$|z| = r$ $\quad \varphi(z) = 0 \quad \ln\frac{|z|}{r} = 0$

$|\varphi(z)| \le M$. $\quad z \in \overline{B}(0;r) \setminus \{0\}$.

$|z| = \delta << r$. Small enough.

s.t. $\varphi(z) + \varepsilon \ln \frac{\delta}{r} \le 0$.

$\therefore$ For any $\varepsilon > 0$.

$$\varphi_\varepsilon(z) = \varphi(z) + \varepsilon \ln \frac{|z|}{r} \le 0, \text{ for } 0 < |z| \le r.$$

$\therefore$ For $0 < |z| \le r$.

let $\varepsilon \to 0^+$

$\therefore \varphi(z) \le$

$\therefore$ For $0 < |z| \le r$. $\quad z \ne 0$

let $\varepsilon \to 0^+$

$\therefore \varphi(z) \le 0 \Rightarrow \begin{matrix} h \le u \\ u \le h \end{matrix} \Rightarrow h = u,\ 0 < |z| \le r$

redefine $u(0) = h(0)$.

0. for $0 < |z| \le r$.