COMPLEX ANALYSIS

ASSIGNMENT VI; DUE JUNE 7, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

51. Given an entire function which is real on the real axis and imaginary on the imaginary axis, prove that it is an odd function, i.e., $f(z) = -f(-z)$.

52. Characterize $\mathrm{Aut}(D)$, where $D = U \setminus \{(\frac{1}{3}, 0), (\frac{-1}{4}, 0)\}$.

53. Let $a > 0$, and define $\mathcal{C} = \{z \in \mathbb{C} \mid \frac{|z-a|}{|z+a|} = \rho\}$, for some $\rho > 0$. Show that (i) $\mathcal{C}$ is a circle, and (ii) a and $-a$ are symmetric with respect to $\mathcal{C}$.

54. Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ if and only if $T(z) = \frac{a}{z}$ for some nonzero a in $\mathbb{C}$.

55. Let $a, b \in \mathbb{R}$, $a < b$, and let $S = \{x+iy \mid a < x < b \text{ and } y > 0\}$. Find a biholomorphic mapping f from S onto U.

56. Find a holomorphic function that maps the complement of the line segment $[-1.1]$ into the open unit disc U.

57. Show that $g(z) = 2z + \frac{1}{z}$ maps the exterior of the unit circle conformally onto the exterior of the ellipse: $\frac{x^2}{9} + y^2 = 1$.

58. Let Ω be an open connected set bounded by two circles which are tangent to each other at one common boundary point a. Map Ω conformally onto the open unit disc U.

59. Let Ω be a domain bounded by two circles $C_1 = \{z \mid |z| = 1\}$ and $C_2 = \{z \mid |z - \frac{1}{2}| = \frac{1}{4}\}$. Map Ω conformally onto an annulus.

60. Let $|a| < R$. Show that

$$f(z) = \frac{R(z-a)}{R^2 - \bar{a}z}$$

is holomorphic on $|z| \le R$, and maps the circle $|z| = R$ into the unit circle.