COMPLEX ANALYSIS

ASSIGNMENT V; DUE MAY 31, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

41. Let u be a bounded real-valued harmonic function in U. Is the conjugate harmonic of u also bounded in U? Prove it or give a counterexample.

42. Let f be an entire function, and let R be a rectangle. If $f(R)$ is also a rectangle, prove that f is linear.

43. If g is an entire function with the property that $|g(z)| \to \infty$ as $|z| \to \infty$, prove that $g(\mathbb{C}) = \mathbb{C}$.

44. Suppose $f \in \mathcal{O}(U)$, and there is a constant $M > 0$ such that $|f^{(k)}(0)| \leq M^k$ for all k. Show that f can be extended holomorphically to an entire function.

45. Let f be holomorphic and bounded on $|z + i| > \frac{1}{2}$ and real on $(-1, 1)$. Show that f is a constant function.

46. Characterize $\mathrm{Aut}(U \setminus \{0\})$.

47. Find the image of $\Omega = \{z \in \mathbb{C} \mid \mathrm{Re}z < 0,\ |\mathrm{Im}z| < \pi\}$ under the exponential function.

48. Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a normal family of holomorphic functions on a domain D. Prove that $\{f'_\alpha\}_{\alpha \in \Lambda}$ is a normal family.

49. Let $\mathcal{F}$ be the class of all $f \in \mathcal{O}(U)$ such that $\mathrm{Re}f > 0$ and $f(0) = 1$. Is $\mathcal{F}$ a normal family?

50. Let $\mathcal{F}$ be the class of all $f \in \mathcal{O}(U)$ for which

$$\iint_U |f(z)|^2 dxdy \leq 1.$$

Is $\mathcal{F}$ a normal family?