COMPLEX ANALYSIS

ASSIGNMENT IV; DUE MAY 3, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

31. Suppose that g is entire and $g(z)$ is real if and only if z is real. Show that g can have at most one zero.

32. Suppose $f \in \mathcal{O}(U)$. Prove that there is a sequence $\{z_n\}$ in U such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.

33. Let $f \in \mathcal{O}(\bar{U})$. If $|f(z)| \leq 1$ for $|z| = 1$ and $f(0) = \frac{1}{2}$. Show that $|f(z)| \leq \frac{3|z|+1}{2}$ for all $z \in U$.

34. Let $f \in \mathcal{O}(\bar{U})$, and $|f(z)| < 1$ for $|z| = 1$. How many fixed points must f have in the disc?

35. Let K be a compact set in $\mathbb{C}$, and let $g(z) = az + 1$, $a \in \mathbb{C}$. Suppose that the origin lies in the convex hull of K(the smallest closed convex set containing K). Show that $\sup_{z \in K}|g(z)| \geq 1$. Note that 0 may not lie in K.

36. Find holomorphic function $f(z)$ whose real part is $x^3 - 3x^2y - 3xy^2 + y^3$.

37. Write out Poisson's integral formula on a disc $D(a; R) = \{z \mid |z - a| < R\}$, if u is continuous on $\bar{D}(a; R)$ and harmonic in $D(a; R)$.

38. Let Ω be a region, and let $f_n \in \mathcal{O}(\Omega)$, $n \in \mathbb{N}$. Suppose u_n is the real part of f_n and $\{u_n\}$ converges uniformly on compact subsets of Ω and $\{f_n(z)\}$ converges for at least one point in Ω. Prove that $\{f_n\}$ converges uniformly on compact subsets of Ω.

39. If u is real-valued and harmonic on a connected open set, and if u^2 is also harmonic, prove that u is a constant function.

40. Show that the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, when written in polar coordinates, takes the form

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$