COMPLEX ANALYSIS

ASSIGNMENT III; DUE APRIL 26, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

21. Evaluate $\int_0^\infty \frac{x^\lambda}{a^2+x^2}dx, -1 < \lambda < 1, a > 0$.

22. Evaluate $\int_0^\infty \frac{\ln x}{x^\lambda(1+x)}dx, 0 < \lambda < 1$.

23. Let f be a holomorphic function defined on the open unit disc such that $|f(\frac{1}{n})| \leq \frac{1}{3^n}$ for $n \geq 2$. Prove that f is identically zero.

24. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial with all a_j real and $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$. Show that all of the zeros of $p(z)$ lie inside the closed unit disc.

25. Let $p(z) = 1 + 2z - 18z^4$. Show that all the zeros of p lie within the open disc $D = (0; \frac{2}{3})$.

26. Show that the only univalent entire functions are the affine functions $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$.

27. Suppose g is holomorphic in the punctured plane $z \neq 0$ and satisfies $|g(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$. Prove g is a constant.

28. Let $\{m_1, m_2, \cdots, m_k\}$ be a set of positive integers and

$$R(z) = \frac{1}{(z^{m_1}-1)(z^{m_2}-1)\cdots(z^{m_k}-1)}.$$

Find the coefficient c_{-k} in the Laurent expansion for $R(z)$ about the point $z = 1$.

29. Let $g \in \mathcal{O}(\Omega)$, where $\Omega = U \setminus \{0\}$. Suppose that $\iint_\Omega |g(z)|^2 dxdy < \infty$. Show that 0 is a removable singularity of g.

30. Show that the converse of Darboux-Picard's theorem is false: Find a simple closed curve C and a function f which is holomorphic on and inside C such that f is univalent inside C but not on C.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX