COMPLEX ANALYSIS

ASSIGNMENT II; DUE MARCH 22, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

11. Apply the Residue Theorem to evaluate $\int_0^\infty \frac{dx}{1+x^m}$, where $m \geq 2$ is a positive integer. (Hint: Use the boundary of a sector with angle $\frac{2\pi}{m}$ as the contour.)

12. Apply the Residue Theorem to evaluate $\int_0^\infty \frac{x^2}{1+x^{10}} dx$. (See the hint of exercise 11.)

13. Apply the Residue Theorem to evaluate $\int_0^\infty \frac{\sin^2 x}{x^2} dx$. (Hint: consider $\frac{e^{2iz}-1-2iz}{z^2}$.)

14. Suppose f is an entire function and $|f(z)| = 1$ on $|z| = 1$. Prove that $f(z) = e^{i\theta_0} z^m$, for some $\theta_0 \in \mathbb{R}$ and $m \in \mathbb{N}$.

15. Show that if f is meromorphic in the extended plane, then f is a rational function.

16. Suppose that f is holomorphic inside and on a simple closed curve C and has no zeros on C. Show that, if m is a positive integer, then

$$\frac{1}{2\pi i} \int_C z^m \frac{f'(z)}{f(z)} dz = \sum_k z_k^m,$$

where the sum is taken over all the zeros $\{z_k\}$ of f inside C.

17. Show that there is no non-constant holomorphic function in the open unit disc which is continuous up to the unit circle and is real on the unit circle.

18. Suppose f is an entire function such that $f(0) = 0$ and $\text{Re} f(z) \to 0$ as $|z| \to \infty$. Prove that $f \equiv 0$.

19. Let Ω be a bounded domain containing 0 in $\mathbb{C}$, and let f be a holomorphic function on Ω that maps Ω into itself. Suppose also that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) \equiv z$.

20. Is it possible to approximate $1/z$ uniformly by polynomials on the annulus $A = \{z \mid \frac{1}{2} < |z| < 1\}$? Show your reason.

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