COMPLEX ANALYSIS

ASSIGNMENT I; DUE MARCH 15, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

1. Show that the series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ converges on $\{|z| \leq 1\}$ except at $z = 1$.

2. Suppose that f is holomorphic in a region and that, at every point, either $f = 0$ or $f' = 0$. Show that f is a constant.

3. Prove that a nonconstant holomorphic function cannot map an open region into a straight line or into a circular arc.

4. Let $U = \{z \in \mathbb{C} | |z| < 1\}$ be the open unit disc. For every $a \in U$, define $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ on U. Show that $\varphi_a \in \mathrm{Aut(U)}$, the automorphism group of U, i.e. an automorphism of U is a holomorphic map from U into itself which is one-to-one and onto. Show also that φ_a maps ∂U one-to-one and onto ∂U.

5. Let $g(z)$ be an entire function with $\mathrm{Im} g(z) \leq 0$. Show that g is a constant function.

6. Suppose that f is an entire function satisfying $|f(z)| \leq \frac{1}{|\mathrm{Im} z|}$ for all z. Prove that $f \equiv 0$.

7. Suppose $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is bounded by 1 for $|z| \leq 1$. Show that $|P(z)| \leq |z|^n$ for all $|z| \geq 1$.

8. Let g be an entire function such that $|g(z)| \leq A + B|z|^k$, where $k > 0$, $A > 0$, $B > 0$. Show that g is a polynomial with degree less than or equal to k.

9. Find the sum of the distances from the point 1 to the other nth roots of 1. Divide the result by n and let $n \to \infty$ to conclude that the average distance from 1 to a point on $|z| = 1$ is $4/\pi$.

10. Prove Lagrange's identity:

$$\left| \sum_{k=1}^{n} z_k w_k \right|^2 = \left(\sum_{k=1}^{n} |z_k|^2 \right) \left(\sum_{k=1}^{n} |w_k|^2 \right) - \sum_{k \leq j} |z_k \bar{w}_j - z_j \bar{w}_k|^2.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX