Solution to Midterm Examination No. 1

1. (a) False.
If the truth values of p, q, r, and s are 0, 0, 1, and 0, respectively, then the truth value of $(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$ is 0 and that of $(p \Rightarrow r) \Rightarrow (q \Rightarrow s)$ is 1. Therefore, $(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$ and $(p \Rightarrow r) \Rightarrow (q \Rightarrow s)$ are not logically equivalent.

(b) True.

$$\begin{aligned}\overline{A} \cup \overline{B} \cup (A \cap B \cap \overline{C}) =& \overline{(A \cap B)} \cup (A \cap B \cap \overline{C}) \\ =& (\overline{(A \cap B)} \cup (A \cap B)) \cap (\overline{(A \cap B)} \cup \overline{C}) \\ =& (\overline{(A \cap B)} \cup \overline{C}) \\ =& \overline{A} \cup \overline{B} \cup \overline{C}.\end{aligned}$$

(C) True.
Consider partitioning the set $\{1, 2, ..., 100\}$ into 10 subsets: $\{1\}, \{2, 3, 4\}, \{5, 6, ..., 9\}, \{10, 11, ..., 16\}, \{17, 18, ..., 25\}, \{26, 27, ..., 36\}, \{37, 38, ..., 49\}, \{50, 51, ..., 64\}, \{65, 66, ..., 81\}, \{82, 83, ..., 100\}$. For any two distinct integers x and y in the same subset, we have $0 < |\sqrt{x} - \sqrt{y}| < 1$. If 11 integers are selected from the set $\{1, 2, ..., 100\}$, by the pigeonhole principle, at least two, say x and y, must be in the same subset and thus satisfy $0 < |\sqrt{x} - \sqrt{y}| < 1$.

2. (a) Let $l : B \to A$ with

$$l(u) = 3, l(v) = 1, l(w) = 2, l(x) = 4, l(y) = 1, l(z) = 1.$$

Then for all $a \in A$, $(l \circ f)(a) = a$ and hence l is a left inverse of f.

(b) Let l be a left inverse of f. Then we must have

$$l(u) = 3, l(w) = 2, l(x) = 4, l(z) = 1$$

and $l(v)$ and $l(y)$ can be any arbitrary element in A. So there are $4 \cdot 4 = 16$ possible different left inverses of f.

(c) Since $v \in B$ is not in the range of f, $f(A)$, there does not exist any element $a \in A$ such that $f(a) = v$. Hence for any function $r : B \to A$, $(f \circ r)(v) = f(r(v)) \neq v$. Therefore, f does not have a right inverse.

3. First guess for all $n \in \mathcal{N}$,

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i \right)^2.$$

We then prove this identity by mathematical induction.

Induction basis: For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \left(\sum_{i=1}^{1} i \right)^2 .$$

Induction step: Assume that this is true for $n = k$, i.e.,

$$\sum_{i=1}^{k} i^3 = \left(\sum_{i=1}^{k} i \right)^2 .$$

Then, for $n = k + 1$, recalling that $\sum_{i=1}^{k} i = k(k+1)/2$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\sum_{i=1}^{k} i \right)^2 + (k+1)^3 \\
&= \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{[k^2 + 4(k+1)](k+1)^2}{4} \\
&= \frac{(k+2)^2(k+1)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2} \right]^2 \\
&= \left(\sum_{i=1}^{k+1} i \right)^2 .
\end{aligned}$$

Therefore, by mathematical induction, for all $n \in \mathcal{N}$,

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i \right)^2 .$$

4. (a) The Hasse diagram for a total order on A must be in the form as shown in Fig. 1. Hence there are $5! = 120$ total orders.

(b) For the symmetric condition of an equivalence relation, $(x, y) \in R \Rightarrow (y, x) \in R$. Yet for the antisymmetric condition of a partial order, $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$. Hence for both an equivalence relation and a partial order, we cannot have $(x, y) \in R$ if $x \neq y$. Therefore, the only possibility for R is $\{(a, a), (b, b), (c, c), (d, d), (e, e)\}$.

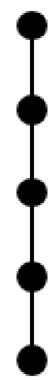

Figure 1: Hasse diagram for a total order.

(c) The corresponding equivalence classes are $\{a\}$,$\{b\}$,$\{c\}$,$\{d\}$, and $\{e\}$.

5. (a) The corresponding zero-one matrix $\boldsymbol{M}$ is

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) d, e, f.

(c) None.

(d) a.

(e) a, b.

(f) f.

6. (a) Note that there should be at least 12 different sets of three jokes. Let n be the number of jokes. We have $\binom{n}{3} \geq 12$ and hence $n \geq 6$. So Professor Chang knows at least 6 jokes.

(b) Since

$$18000 = 2^4 \cdot 3^2 \cdot 5^3$$

the number of (positive) divisors of 18000 is $(4+1) \cdot (2+1) \cdot (3+1) = 60$.

7. (a) By Binomial Theorem, we have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.$$

Then

$$(1+x)^n(1+x)^n = \left[\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right]\left[\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right].$$

On the left-hand side, the coefficient of x^{n+1} in $(1+x)^{2n}$ is $\binom{2n}{n+1}$. On the right-hand side, the coefficient of x^{n+1} is

$$\binom{n}{1}\binom{n}{n}+\binom{n}{2}\binom{n}{n-1}+\cdots+\binom{n}{n}\binom{n}{1} = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n+1-k} = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}.$$

Therefore,

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \binom{2n}{n+1}.$$

(b) Consider that there are n electrical engineering students and n computer science students. We want to select a team of $n+1$ members. There are $\binom{2n}{n+1}$ ways to select $n+1$ members from the $2n$ people, which is the result on the right-hand side of the identity. Another way is to select k members from the electrical engineering students to join the team first and then select $k-1$ computer science students that do not join the team (with the rest computer science students joining the team), for $1 \le k \le n$. Given k, there are $\binom{n}{k}$ ways to select the electrical engineering students and $\binom{n}{k-1}$ ways for the computer science students. Hence the total number of ways is $\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}$, which is exactly the result on the left-hand side of the identity.

8. (a) $c_3 = 3$.

(b) Note that $|S_n| = n!$. By the principle of inclusion and exclusion, we have

$$\begin{aligned} c_n &= |\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_{n-1}}| \\ &= |S_n| - |A_1\cup A_2\cup\cdots\cup A_{n-1}| \\ &= n! - \alpha_1 + \alpha_2 + \cdots + (-1)^{n-1}\alpha_{n-1} \end{aligned}$$

where $\alpha_1 = |A_1|+|A_2|+\cdots+|A_{n-1}|$, $\alpha_2 = |A_1\cap A_2|+|A_1\cap A_3|+\cdots+|A_{n-2}\cap A_{n-1}|$, $\ldots$, $\alpha_{n-1} = |A_1\cap A_2\cap\cdots\cap A_{n-1}|$.

It is clear that $|A_i| = (n-2+1)! = (n-1)!$ as we can consider $i(i+1)$ as one object in the permutation. For $|A_i \cap A_{i+1}|$, we can consider $i(i+1)(i+2)$ as one object in the permutation and thus $|A_i\cap A_{i+1}| = (n-3+1)! = (n-2)!$. For $|A_i\cap A_j|$, where $i<j$ and $j \ne i+1$, we can consider $i(i+1)$ and $j(j+1)$ as two objects in the permutation, and thus $|A_i\cap A_j| = (n-4+2)! = (n-2)!$. Hence $|A_i\cap A_j|$ is always equal to $(n-2)!$, for $1\le i<j\le n-1$.

Similarly, we obtain $|A_{i_1}\cap A_{i_2}\cap A_{i_3}| = (n-3)!$, for $1 \le i_1 < i_2 < i_3 \le n-1$. In general, we have

$$|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_r}| = (n-r)!, \text{ for } 1\le i_1<i_2<\cdots<i_r\le n-1,\ 1\le r\le n-1.$$

Therefore,

$$\begin{aligned} c_n &= n! - \sum_{r=1}^{n-1} (-1)^{r-1} \binom{n-1}{r} (n-r)! \\ &= \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (n-r)!. \end{aligned}$$