EECS 2060 Discrete Mathematics — Chi-chao Chao
Spring 2021

Midterm Examination No. 1
7:00pm to 10:00pm, April 9, 2021

Problems for Solution:

1. (15%) True or false. (You need to justify your answer. If it is true, prove it. Otherwise, disprove it by a counterexample or in other ways.)
   (a) (5%) $(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$ and $(p \Rightarrow r) \Rightarrow (q \Rightarrow s)$ are logically equivalent, where p, q, r, and s are statements
   (b) (5%) $\overline{A} \cup \overline{B} \cup (A \cap B \cap \overline{C}) = \overline{A} \cup \overline{B} \cup \overline{C}$, where A, B, and C are sets.
   (c) (5%) If 11 integers are selected from $\{1, 2, \ldots, 100\}$, then there are at least two, say x and y, such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.
2. (15%) Let $A = \{1, 2, 3, 4\}$ and $B = \{u, v, w, x, y, z\}$. Consider the function $f : A \to B$ with
$$f(1) = z, f(2) = w, f(3) = u, f(4) = x.$$
   (a) (5%) Recall that f is said to have a *left inverse* $l : B \to A$ if
$$(l \circ f)(a) = a, \text{ for all } a \in A.$$
   Does f have a left inverse? If yes, find one. Otherwise, explain why none exists.
   (b) (5%) How many different possible left inverses can f have?
   (c) (5%) The function f is said to have a *right inverse* $r : B \to A$ if
$$(f \circ r)(b) = b, \text{ for all } b \in B.$$
   Does f have a right inverse? If yes, find one. Otherwise, explain why none exists.
3. (10%) Notice that
$$\begin{aligned} 1 &= 1^2 \\ 1+8 &= (1+2)^2 \\ 1+8+27 &= (1+2+3)^2 \\ 1+8+27+64 &= (1+2+3+4)^2 \\ 1+8+27+64+125 &= (1+2+3+4+5)^2. \end{aligned}$$
   Guess the general pattern, i.e., give an identity with $n \in \mathcal{N}$ involved in two sides, and prove it.

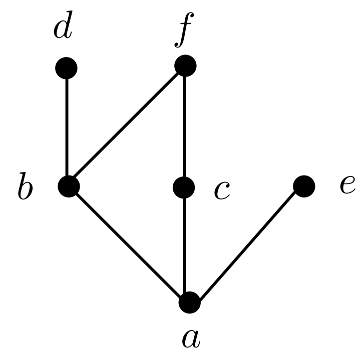
4. (15%) Let $A = \{a, b, c, d, e\}$.

   (a) (5%) Among all the relations on A, how many are total orders?

   (b) (5%) If a relation R on A is both an equivalence relation and a partial order, list all the ordered pairs in R.

   (c) (5%) Find the corresponding equivalence classes for the relation R in (b).

5. (15%) Let $A = \{a, b, c, d, e, f\}$. Consider the partial order R on A with Hasse diagram shown below.



   (a) (5%) Find the corresponding zero-one matrix $\boldsymbol{M}$, where the rows (from top to bottom) and the columns (for left to right) of $\boldsymbol{M}$ are indexed in the order a, b, c, d, e, f.

   (b) (2%) Find the maximal elements.

   (c) (2%) Find a greatest element, if it exists.

   (d) (2%) Find a least element, if it exists.

   (e) (2%) Find all lower bounds of $\{b, f\}$.

   (f) (2%) Find the least upper bound of $\{a, b, c\}$, if it exists.

6. (a) (5%) Professor Chang has taught the same course for the last twelve years and tells three jokes each year. He has never told the same set of three jokes twice (the order of the jokes is unimportant). How many jokes at least must he know?

   (b) (5%) Find the number of (positive) divisors of 18000.

7. (10%) In this problem you are required to prove in two ways that for $n \in \mathcal{N}$,

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n}{n+1}. \qquad (*)$$

   (a) (5%) Use Binomial Theorem and consider $(1+x)^n(1+x)^n$ to show $(*)$.

   (b) (5%) Give a combinatorial argument to show $(*)$. (*Hint:* Count in two ways the number of ways to select a team of $n+1$ members, from a group of n electrical engineering students and n computer science students.)

8. (10%) Recall that a permutation of $\mathcal{N}_n = \{1, 2, \ldots, n\}$ is a bijection from $\mathcal{N}_n$ to $\mathcal{N}_n$. A permutation can be represented by the resulting sequence, e.g., the permutation π of $\mathcal{N}_5$ with $\pi(1) = 4, \pi(2) = 3, \pi(3) = 1, \pi(4) = 5, \pi(5) = 2$ can be represented by the sequence $4\,3\,1\,5\,2$. Let C_n denote the set of the permutations of $\mathcal{N}_n$ in which no two adjacent integers are consecutive in the resulting sequences; in other words, the $n-1$ patterns $1\,2$, $2\,3$, $3\,4$, $\ldots$, $(n-1)\,n$ should not appear in (the sequences of) the permutations. For example, $4\,3\,1\,5\,2, 5\,4\,3\,2\,1$ are in C_5, but $3\,1\,2\,5\,3, 5\,2\,3\,4\,1$ are not. Let c_n be the cardinality of C_n, i.e., $c_n = |C_n|$.

   (a) (2%) Find c_3.

   (b) (8%) Find a formula for c_n, $n \geq 2$. (*Hint:* Let S_n denote the set of all permutations of $\mathcal{N}_n$. Define $A_i = \{\pi \in S_n : \text{the pattern } i\,(i+1) \text{ appears in } \pi\}$, for $i = 1, 2, \ldots, n-1$. Express C_n in terms of A_i, $i = 1, 2, \ldots, n-1$.)