Angular Momentum

The angular momentum of a particle is

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y,\ zp_x - xp_z,\ xp_y - yp_x)$$

Making the substitutions $p_\alpha = -i\hbar \frac{\partial}{\partial x_\alpha}$, the quantum version of $\vec{L}$ takes the following form

$$\boxed{\begin{aligned} L_x &= -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \\ L_y &= -i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \\ L_z &= -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \end{aligned}}$$

Note Take L_z as an example, $[x, \frac{\partial}{\partial y}] = 0$ and $[y, \frac{\partial}{\partial x}] = 0$. So, it is ok to change the "order": $x\frac{\partial}{\partial y} = \frac{\partial}{\partial y}x$

OR $y\frac{\partial}{\partial x} = \frac{\partial}{\partial x}y$

Commutators of L_x, L_y, L_z

Let's compute the commutator $[L_x, L_y]$. There are many ways to proceed. Here I would like to use commutator algebra :)

$$\boxed{[L_x, L_y]} = [yP_z - zP_y,\ zP_x - xP_z]$$

$$= [yP_z, zP_x] - \cancel{[yP_z, xP_z]}^{\,0}$$

$$- \cancel{[zP_y, zP_x]}_{\,0} + [zP_y, xP_z]$$

x, y, P_z commute!

z, P_x, P_y commute!

$$= y[P_z, zP_x] + \cancel{[y, zP_x]}^{\,0} P_z$$

$$+ z\cancel{[P_y, xP_z]}^{\,0} + [z, xP_z] P_y$$

$$= y[P_z, z]P_x + yz\cancel{[P_z, P_x]}^{\,0}$$

$[z, P_z] = i\hbar$

$$+ x[z, P_z]P_y + \cancel{[z, x]}^{\,0} P_z P_y$$

Finally, $[L_x, L_y] = y(-i\hbar)P_x + x(i\hbar)P_y$

$$= i\hbar(xP_y - yP_x)$$

$$= \boxed{i\hbar L_z}$$

Similarly, one can work out other commutators

$$\boxed{\begin{aligned}[L_x, L_y] &= i\hbar L_z \\ [L_z, L_x] &= i\hbar L_y \\ [L_y, L_z] &= i\hbar L_x\end{aligned}}$$

It is quite surprising that these commutators lead to quantization of angular momentum !

Introduce the Hermitian operator L^2 :

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

It's straightforward to show that L^2 commute with all components L_x, L_y, L_z.

$$\boxed{[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0}$$

Let's compute $[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z]$

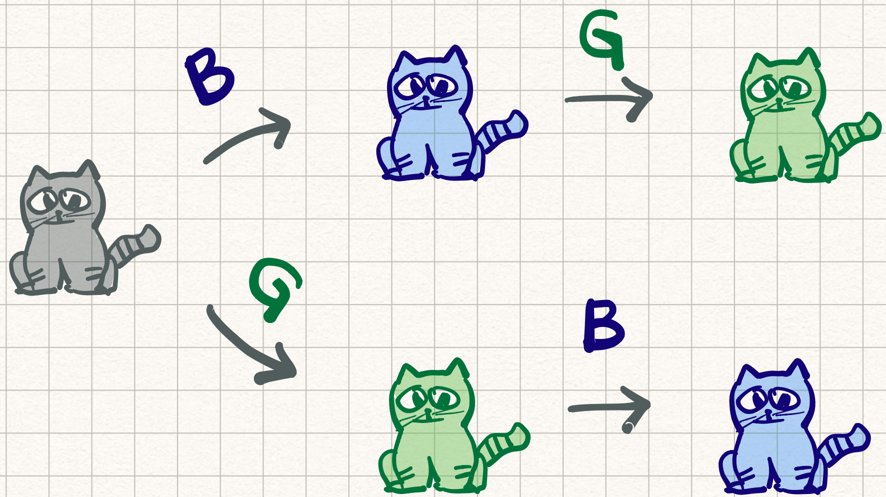
$$= [L_x^2, L_z] + [L_y^2, L_z] + \cancel{[L_z^2, L_z]}^{\,0}$$

$$= L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y$$

$$= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y = 0 \,!$$

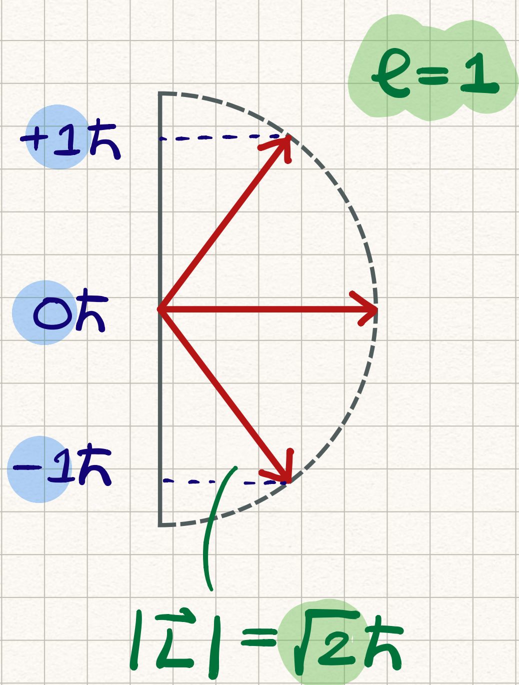
Strange Quantization Rules

Suppose two operators B, G with $[B, G] \neq 0$.



$$BG - GB \neq 0$$

Choose the maximally commuting set : L^2, L_z



① magnitude of $\vec{L}$

$$|\vec{L}| = \sqrt{\ell(\ell+1)}\,\hbar$$

$$\ell = 0, 1, 2, 3, \cdots$$

② z-comp. of $\vec{L}$

$$L_z = m\hbar$$

$$m = -\ell, -(\ell-1), \cdots, 0, \cdots, \ell$$

$(2\ell+1)$ states in total.

Ladder Operators

Define a set of ladder operators U, D

$$\boxed{\begin{aligned} U &= L_x + iL_y \\ D &= L_x - iL_y \end{aligned}}$$

Because $\vec{L}$ is Hermitian,

$$U^\dagger = L_x^\dagger - iL_y^\dagger = L_x - iL_y = D$$

$$D^\dagger = L_x^\dagger + iL_y^\dagger = L_x + iL_y = U$$

① $UD = (L_x + iL_y)(L_x - iL_y)$

$$= L_x^2 + L_y^2 + i(L_yL_x - L_xL_y)$$

$[L_y, L_x] = -i\hbar L_z$

$$= L^2 - L_z^2 + \hbar L_z$$

② $DU = (L_x - iL_y)(L_x + iL_y)$

$$= L_x^2 + L_y^2 - i(L_yL_x - L_xL_y)$$

$[L_y, L_x] = -i\hbar L_z$

$$= L^2 - L_z^2 - \hbar L_z$$

③ $\underline{[L_z, U]} = [L_z, L_x] + i[L_z, L_y]$

$$= i\hbar L_y + \hbar L_x = \hbar U$$

similarly $\underline{[L_z, D]} = -\hbar D$

Up and Down the Ladder

Choose to diagonalize L^2, L_z simultaneously

$$L^2 |a,b\rangle = a\hbar^2 |a,b\rangle$$

$$L_z |a,b\rangle = b\hbar |a,b\rangle$$

Construct another state by the ladder op. U

$$|\psi\rangle = U|a,b\rangle$$

We would like to show that $|\psi\rangle = |a, b+1\rangle$

$\oslash$ $[L^2, L_\alpha] = 0 \longrightarrow [L^2, U] = 0$

$$L^2|\psi\rangle = L^2 U|a,b\rangle = UL^2|a,b\rangle$$

$$= U(a\hbar^2)|a,b\rangle$$

$$= (a\hbar^2)U|a,b\rangle = a\hbar^2|\psi\rangle$$

YES! $|\psi\rangle$ is also an eigenstate of L^2 with the same eigenvalue $a\hbar^2$!

$$[L_z, U] = \hbar U \rightarrow L_z U = U(L_z + \hbar)$$

$$L_z |\psi\rangle = L_z U |a,b\rangle = U(L_z + \hbar)|a,b\rangle$$

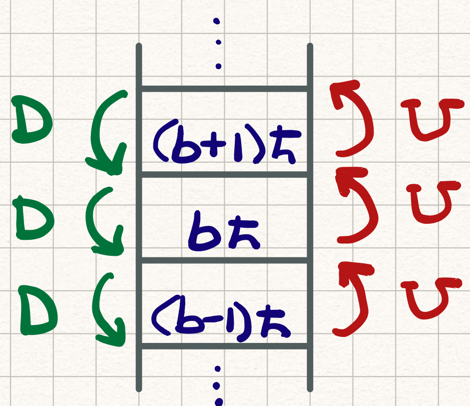
$$= U\,(b+1)\hbar\,|a,b\rangle = (b+1)\hbar\, U|a,b\rangle$$

$$= (b+1)\hbar\,|\psi\rangle$$

YES! $|\psi\rangle$ is also an eigenstate of L_z with the eigenvalue $(b+1)\hbar$!

One can also construct the state $|\phi\rangle = D|a,b\rangle$ and show that ~

$$|\phi\rangle = |a, b-1\rangle.$$



$a\hbar^2$ remains the same

One can use ladder operators U, D to construct eigenstates with the same $a\hbar^2$ but different $(b\pm1)\hbar$

Quantization of Angular Momentum

The "ladder" of L_z eigenvalues has ends.

$$L^2 = \underbrace{L_x^2 + L_y^2}_{\text{positive definite}} + L_z^2 \longrightarrow \boxed{a\hbar^2 \geq (b\hbar)^2}$$

Suppose the maximal value is $\boxed{b_{max} = \ell}$

$$\longrightarrow \quad \ell^2 \leq a \quad \text{but} \quad (\ell+1)^2 > a$$

Let us apply the ladder op. U on $|a,\ell\rangle$

$$L_z \underline{U|a,\ell\rangle} = (\ell+1)\hbar \, \underline{U|a,\ell\rangle}$$

This means that $U|a,\ell\rangle = |a,\ell+1\rangle$, violating ℓ is the maximum of L_z eigenvalue... The only resolution is $\boxed{U|a,\ell\rangle = 0}$!

$$DU|a,\ell\rangle = (L^2 - L_z^2 - \hbar L_z)|a,\ell\rangle = 0$$

$$\longrightarrow \quad a\cancel{\hbar^2} - \ell^2\cancel{\hbar^2} - \ell\cancel{\hbar^2} = 0$$

$$\boxed{a = \ell^2 + \ell = \ell(\ell+1)}$$

relation between a and $b_{max} = \ell$.

Similarly, the eigenvalue of L_z should have a minimum value $b_{min} = \ell - n$.

Here n is a non-negative integer.

Construct the state $D|a, \ell-n\rangle$

$$L_z D|a,\ell-n\rangle = (\ell-n-1)\hbar\, D|a,\ell-n\rangle$$

But this cannot be true unless the ladder op. D kills the state $|a,\ell-n\rangle$

$$D|a,\ell-n\rangle = 0 \quad !$$

$$UD|a,\ell-n\rangle = (L^2 - L_z^2 + \hbar L_z)|a,\ell-n\rangle = 0$$

$\ell(\ell+1)$ —
$$a\cancel{\hbar^2} - (\ell-n)^2\cancel{\hbar^2} + (\ell-n)\cancel{\hbar^2} = 0$$

$$\cancel{\ell^2} + \ell - \cancel{\ell^2} - n^2 + 2\ell n + \ell - n = 0$$

$$2\ell + 2\ell n = n + n^2 \rightarrow 2\ell(\cancel{n+1}) = n(\cancel{n+1})$$

$$\rightarrow \ell = \frac{n}{2}$$

angular momentum quantization

$$\boxed{\begin{aligned} L^2 &\rightarrow a\hbar^2 = \ell(\ell+1)\hbar^2 \\ L_z &\rightarrow b\hbar = \ell\hbar, (\ell-1)\hbar, \dots, -\ell\hbar \end{aligned}}$$