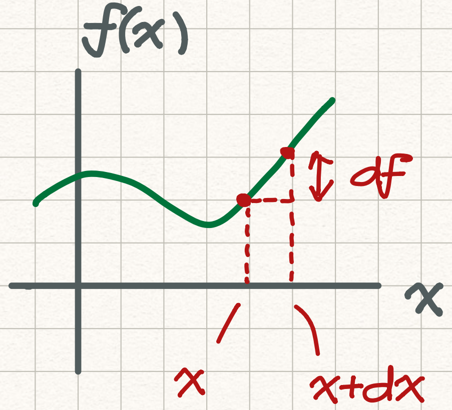
Partial Derivative

Let's review the notion of differential d



From Δ to d

$$\Delta f \equiv f(x+\Delta x) - f(x)$$

$$df \equiv f(x+dx) - f(x)$$

According to calculus, it is easy to show that

$$df = f(x+dx) - f(x) = \frac{f(x+dx) - f(x)}{dx}\, dx$$

$$\rightarrow \quad \boxed{df = \frac{df}{dx}\, dx}$$

The notion of differential can be readily generalized to the multi-variable functions.

$$df \equiv f(x+dx,\, y+dy) - f(x, y)$$

This is called the total differential of f.

partial derivative

When computing total differential, one finds useful to define partial derivative:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{f(x+dx, y) - f(x,y)}{dx}$$

Similarly, one can define the partial derivative with respect to the other variable y:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \frac{f(x, y+dy) - f(x,y)}{dy}$$

What about the 2^{nd} partial derivatives?

Taking partial derivatives of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ ~

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

are they the same?

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Skipping the proof, as long as the 2^{nd} partial derivative is continuous,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

example Find 1^{st} and 2^{nd} partial derivatives of the function $f(x,y) = \cos(xy) + y^2$

$$\frac{\partial f}{\partial x} = -y\sin(xy) \longrightarrow \frac{\partial^2 f}{\partial x^2} = -y^2\cos(xy)$$

$$\frac{\partial f}{\partial y} = -x\sin(xy) + 2y \longrightarrow \frac{\partial^2 f}{\partial y^2} = -x^2\cos(xy) + 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\{-y\sin(xy)\} = -\sin(xy) - xy\cos(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\{-x\sin(xy) + 2y\} = -\sin(xy) - xy\cos(xy)$$

It's clear that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

The order of taking partial derivatives doesn't matter

total differential

$$df = f(x+dx,\, y+dy) - f(x,y)$$

$$= f(x+dx,\, y+dy) - f(x,\, y+dy)$$

$$+ f(x,\, y+dy) - f(x,y)$$

Thus, the total differential df can be written in the following form.

$$df = \frac{f(x+dx,\, y+dy) - f(x,\, y+dy)}{dx}\, dx + \frac{f(x,\, y+dy) - f(x,y)}{dy}\, dy$$

$$\rightarrow \quad \boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

example $\quad f(x,y) = x^2 - y^2$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x\,dx - 2y\,dy \quad \#$$

What if x, y are not independent variables?

Does the relation $df = f_x dx + f_y dy$ remain true?

example $f(x,y) = x^2 - y^2$ and $y = x^2$

$$f(x, y(x)) = x^2 - (x^2)^2 = x^2 - x^4.$$

$$df = 2x\,dx - 4x^3\,dx = (2x - 4x^3)\,dx$$

Or, if one trusts the partial derivatives ☺

$$df = 2x\,dx - 2y\,dy \quad — \quad dy = 2x\,dx$$

$$= 2x\,dx - 2x^2 \cdot 2x\,dx$$

$$= (2x - 4x^3)\,dx$$

From this simple yet inspiring example, one should know that

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

is always true and does not rely on the mutual independence of the variables!

exact & inexact differentials

Sometimes, we are given the following differential

$$A(x,y)\,dx + B(x,y)\,dy \overset{?}{=} df(x,y)$$

and wonder whether it can be written as the total differential of some fn $f(x,y)$.

If YES, it is called exact differential.

If Nope, it is called inexact differential.

Consider the total differential as below:

$$df = A(x,y)\,dx + B(x,y)\,dy$$

It's obvious that $A = \frac{\partial f}{\partial x}$ and $B = \frac{\partial f}{\partial y}$

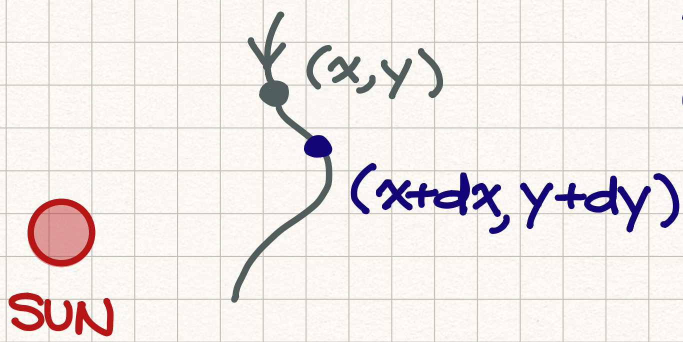
From the relation $\quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$$\Rightarrow \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

It turns out that the above criterion is the necessary and sufficient condition for the differential to be exact

This is quite often encountered in field theory.



The work done by the external force is

$$F_x dx + F_y dy$$

If it is an exact differential, the work done by the force can be written as

$$F_x dx + F_y dy = \vec{F} \cdot d\vec{r} = -dU$$

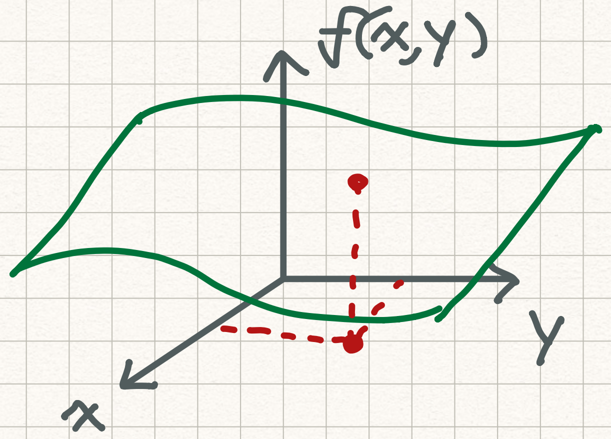
The force can be described by a potential U.

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \longrightarrow \text{exact differential}$$

The potential exists and the force field is called "conservative".

Different perspectives of $f(x,y)$

For a given fn $f(x,y)$, it is natural to view it as a 2D surface. One can introduce an additional variable z



$$z = f(x,y)$$

Thus, there are many ways to look at the relation.

(1) $x = x(y,z)$

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

(2) $y = y(x,z)$

$$dy = \left(\frac{\partial y}{dx}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz$$

Reciprocity relation: $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial y}{\partial x}\right)_z^{-1}$

set $dz = 0$

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy$$

$$dy = \left(\frac{\partial y}{\partial x}\right)_z dx$$

multiply both equations together...

$$\cancel{dx}\,dy = \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z dy\,\cancel{dx}$$

$$\rightarrow \left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial x}\right)_z = 1$$

Cyclic Relation : $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$

set $dx = 0$ $\qquad 0 = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$

$$dy = 0 + \left(\frac{\partial y}{\partial z}\right)_x dz$$

$$\rightarrow \left\{ \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x + \left(\frac{\partial x}{\partial z}\right)_y \right\} dz = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -\left(\frac{\partial x}{\partial z}\right)_y$$

making use of the relation $\left(\frac{\partial x}{\partial z}\right)_y = \left(\frac{\partial z}{\partial x}\right)_y^{-1}$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

YES !

Taylor Series

The trick is similar to the single-variable fn but the algebra is more complicated....

$$f(x,y) = a_0 + \{a_{1x}x + a_{1y}y\} + \{a_{2xx}x^2 + a_{2yy}y^2 + a_{2xy}xy\} + \cdots$$

The task is to find out all coefficients!

(1) zero-th order just plug in $x=0, y=0$

$$f(0,0) = a_0$$ checked ✓

(2) first order taking $\frac{\partial}{\partial x}$ first

$$\frac{\partial f}{\partial x}(x,y) = a_{1x} + \{2a_{2xx}x + a_{2xy}y\} + \cdots$$

plug in $x=0, y=0$ later $\quad \frac{\partial f}{\partial x}(0,0) = a_{1x}$

similarly, one can compute a_{1y}

$$a_{1y} = \frac{\partial f}{\partial y}(0,0)$$

(3) second order

$$\frac{\partial^2 f}{\partial x^2} = (2!)\, a_{2xx} + \cancel{\{\cdots\}}$$ zero @ $x=0, y=0$

$$\frac{\partial^2 f}{\partial y^2} = (2!)\, a_{2yy} + \cancel{\{\cdots\}}$$ zero @ $x=0, y=0$

$$\frac{\partial^2 f}{\partial x \partial y} = a_{2xy} + \cancel{\{\cdots\}}$$ zero @ $x=0, y=0$

Collect all results together ~

@ $x=0, y=0$ implicitly ⍨

$$f(x,y) = f(0,0) + \left\{ \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y \right\} + \frac{1}{2!} \left\{ \frac{\partial^2 f}{\partial x^2} x^2 + 2 \frac{\partial^2 f}{\partial x \partial y} xy + \frac{\partial^2 f}{\partial y^2} y^2 \right\} + \cdots$$

$$= f(0,0) + \sum_i \frac{\partial f}{\partial x_i} x_i + \frac{1}{2!} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + \cdots$$

OR, one can write the Taylor series in the rather interesting form

$$\frac{\partial^2 f}{\partial x^2} x^2 + 2\frac{\partial^2 f}{\partial x \partial y} xy + \frac{\partial^2 f}{\partial y^2} y^2$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f$$

Thus, Taylor Series can be written as

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f \Big|_{0,0}$$

The generalization is now clear,

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^n f \Big|_{x_0,y_0}$$

here $\Delta x = x - x_0$ and $\Delta y = y - y_0$