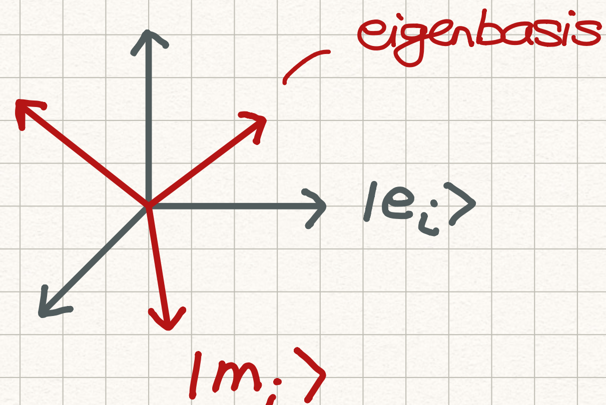
Matrix Diagonalization

For a real and symmetric matrix M, its eigenvectors form an orthonormal basis.

In the eigenbasis, the matrix M is diagonal.



$$M_{ij} = \langle e_i | M | e_j \rangle$$

Note that M can be written as

$$M = \sum_i m_i |m_i\rangle\langle m_i|$$

The matrix representation of the operator M in the eigenbasis is

$$D_{ij} = \langle m_i | M | m_j \rangle$$

$$= \langle m_i | \left(\sum_k m_k |m_k\rangle\langle m_k| \right) | m_j \rangle$$

$$= \sum_k m_k \underbrace{\langle m_i | m_k \rangle}_{\delta_{ik}} \underbrace{\langle m_k | m_j \rangle}_{\delta_{kj}}$$

$$\rightarrow \quad \boxed{D_{ij} = m_i \, \delta_{ij}} \quad \text{— diagonal}$$

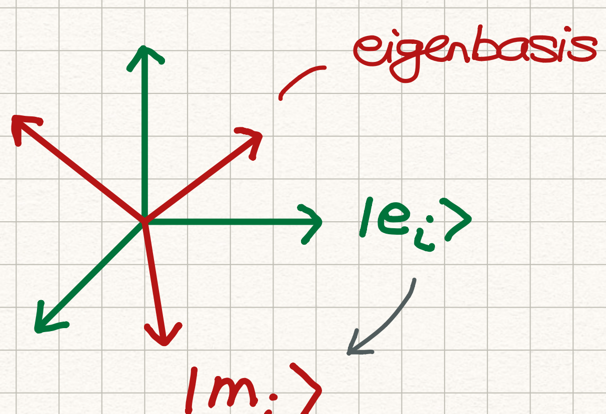
Similarity Transformation

example $M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$

$m_1 = 1$ $\qquad\qquad$ $m_2 = 6$

$|m_1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\qquad\qquad$ $|m_2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

We would like to construct the similarity transformation S that relates M_{ij} & D_{ij}.



$$M_{ij} = \langle e_i | M | e_j \rangle = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

$$D_{ij} = \langle m_i | M | m_j \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

These two bases are related by S:

$$|m_j\rangle = S\,|e_j\rangle$$

matrix elements $\quad S_{ij} = \langle e_i | S | e_j \rangle = \langle e_i | m_j \rangle$

It is then straightforward to derive the similarity transformation:

$$\boxed{S^{-1} M S = D}$$

Use the bra-ket notation to prove the relation

$$(S^{-1}MS)_{ij} = \sum_{k\ell} (S^{-1})_{ik} M_{k\ell} S_{\ell j}$$

$$= \sum_{k\ell} \underbrace{\langle m_i | e_k \rangle \langle e_k}_{\sum_k |e_k\rangle\langle e_k| = \mathbb{1}} | M | \underbrace{e_\ell \rangle \langle e_\ell}_{\sum_\ell |e_\ell\rangle\langle e_\ell| = \mathbb{1}} | m_j \rangle$$

$$= \langle m_i | M | m_j \rangle = m_i \, \delta_{ij} = D_{ij}$$

Let's construct the matrix S_{ij} explicitly,

$$\boxed{S_{ij} = \langle e_i | m_j \rangle} = \begin{pmatrix} |m_1\rangle & |m_2\rangle \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad - \det S = \left(\frac{1}{\sqrt{5}}\right)^2 \cdot 5 = 1$$

One can work out the inverse $(S^{-1})_{ij}$

$$(S^{-1})_{ij} = \frac{1}{\det S} \, C_{ji}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

It is the transpose!

$S^{-1} = S^T$ ☺

(1) $S^{-1} S = \mathbb{1}$

$$(S^{-1} S)_{ij} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ij}$$

(2) $S^{-1} M S = D$

$$(S^{-1} M S)_{ij} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \cdot 1 & 6 \cdot -2 \\ 1 \cdot 2 & 6 \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Diagonal Matrix

The algebra for diagonal matrices is simple.

$$D_{ij} = d_i \delta_{ij}$$

$$D^2_{ij} = \sum_k D_{ik} D_{kj} = \sum_k d_i \overset{d_i \delta_{ij}}{\delta_{ik} d_k} \delta_{kj} = d_i^2 \delta_{ij}$$

It is easy to show that $(D^n)_{ij} = d_i^n \delta_{ij}$

example $M = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$, $M^n = ?$

starting from $S^{-1} M S = D \rightarrow M = S D S^{-1}$

$$M^n = \underbrace{M \cdot M \cdots M}_{n \text{ times}} = (SD\cancel{S^{-1}})(\cancel{S}D\cancel{S^{-1}}) \cdots (\cancel{S}DS^{-1})$$

$$\rightarrow \quad M^n = S\underbrace{(D \cdot D \cdots D)}_{n \text{ times}} S^{-1} = S D^n S^{-1}$$

(1) Diagonalize M → eigenvalues $m = 6, -3, -3$
→ eigenvectors

(2) Construct D from eigenvalues.

(3) Construct S from eigenvectors.

For two diagonal matrices A, B, they commute.

$$[A,B] = AB - BA = 0$$

$$(AB)_{ij} = \sum_k a_i \delta_{ik} b_k \delta_{kj} = a_i b_i \delta_{ij}$$

$$(BA)_{ij} = \sum_k b_i \delta_{ik} a_k \delta_{kj} = a_i b_i \delta_{ij}$$

Applying similarity transformation to A, B:

$$A' = SAS^{-1} \longleftrightarrow A = S^{-1}A'S$$

$$B' = SBS^{-1} \longleftrightarrow B = S^{-1}B'S$$

$$[A',B'] = A'B' - B'A'$$

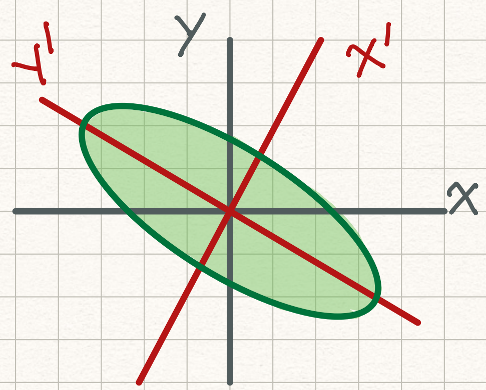
$$= SA\cancel{S^{-1}} \cdot \cancel{S}BS^{-1} - SB\cancel{S^{-1}} \cdot \cancel{S}AS^{-1}$$

$$= SABS^{-1} - SBAS^{-1}$$

$$= S(AB - BA)S^{-1} = 0$$

Thus, $[A,B] = 0$ is independent of the basis choices

Quadratic Curves



Consider the quadratic curve,

$$5x^2 - 4xy + 2y^2 = 30$$

$$\Downarrow$$

$$(x\ y)\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 30$$

In the eigenbasis, the quadratic form is brought into diagonal ~

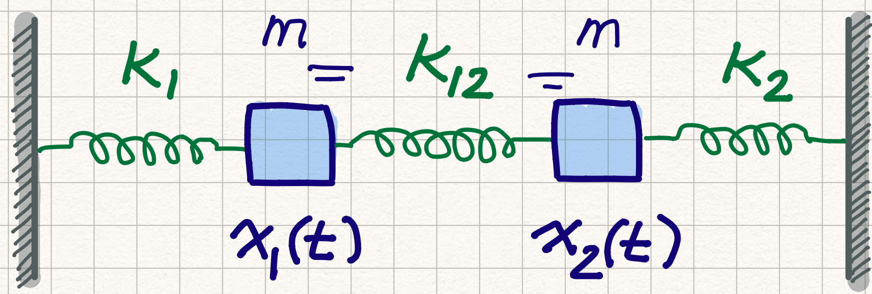
$$(x'\ y')\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 30$$

$$\rightarrow \quad x'^2 + 6y'^2 = 30$$

$$\frac{x'^2}{30} + \frac{y'^2}{5} = 1 \quad \rightarrow \quad a = \sqrt{30},\ b = \sqrt{5}$$

The principal axes are the eigenvectors ☺

Harmonic Oscillators



Write down EOM for the oscillators ~

$$m \frac{d^2x_1}{dt^2} = -k_1 x_1 + k_{12}(x_2 - x_1)$$

$$m \frac{d^2x_2}{dt^2} = -k_2 x_2 - k_{12}(x_2 - x_1)$$

$$\boxed{m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

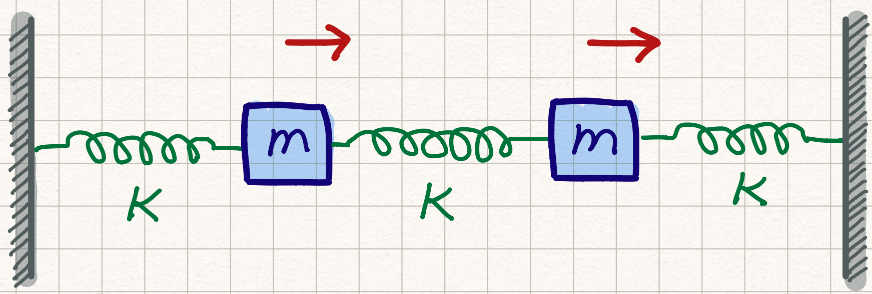
Choose $k_1 = k_2 = k_{12} = k$. The $\mathbb{K}$ matrix becomes

$$\mathbb{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \qquad \begin{vmatrix} 2k-\lambda & -k \\ -k & 2k-\lambda \end{vmatrix} = 0$$

$$(\lambda - 2k)^2 - k^2 = 0 \qquad \lambda = k, 3k$$

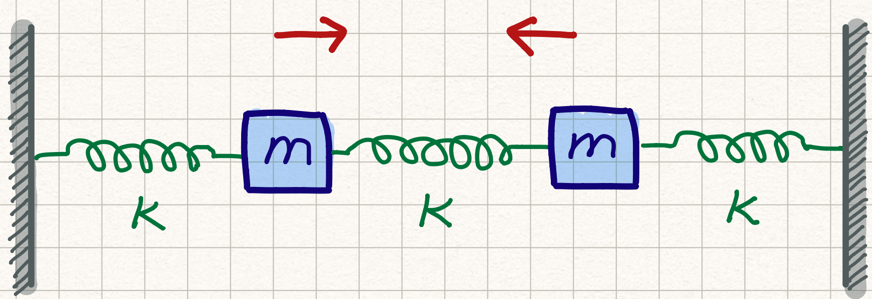
eigenvectors $|k\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad |3k\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Normal Modes for the coupled oscillators



$$\omega_1 = \sqrt{\frac{K}{m}}$$

$$|K\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$\omega_2 = \sqrt{\frac{3K}{m}}$$

$$|3K\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$