Eigenvector & Eigenvalues

A matrix M maps one vector to another:

$$M|x\rangle = |y\rangle$$

Is it possible that M just maps some vector $|m\rangle$ to the same vector $|m\rangle$?

$$M\underbrace{|m\rangle}_{\text{eigenvector}} = \underbrace{m}_{\text{eigenvalue}}|m\rangle$$

eigen = 正

Example $\quad M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ $\quad M|m\rangle = m|m\rangle$

Write down the eigen equation explicitly

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = m\begin{pmatrix} x \\ y \end{pmatrix}$$

A trivial soln is $\underline{(x\ y)^T = (0,0)^T}$. If there are more solutions,

$$\begin{pmatrix} 5-m & -2 \\ -2 & 2-m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \rightarrow \begin{vmatrix} 5-m & -2 \\ -2 & 2-m \end{vmatrix} = 0$$

Some algebra is needed here

$$(5-m)(2-m)-(-2)(-2)=0$$

$$m^2-7m+6=0 \longrightarrow (m-6)(m-1)=0$$

eigenvalues of M : $m = 1, 6$

Let's work out the corresponding eigenvectors.

$m = 1$ case

$y = 2x$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{aligned} 5x-2y &= x \\ -2x+2y &= y \end{aligned}$$

The eigenvector is $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$m = 6$ case

$x = -2y$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{aligned} 5x-2y &= 6x \\ -2x+2y &= 6y \end{aligned}$$

The eigenvector is $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Orthonormal basis

$$\langle 1|6\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$$

orthogonal!

$$= \left(\frac{1}{\sqrt{5}}\right)^2\left[1\cdot(-2)+2\cdot 1\right] = 0$$

The eigenvectors form an orthonormal basis.

$$\langle m|m'\rangle = \delta_{mm'}$$

One can then expand an arbitrary vector $|v\rangle$ in terms of the eigenbasis of M.

$$|v\rangle = \sum_m c_m |m\rangle$$

Let's compute the coefficient c_n

$$\langle n|v\rangle = \sum_m c_m \underbrace{\langle n|m\rangle}_{\delta_{nm}} \rightarrow \boxed{c_n = \langle n|v\rangle}$$

We can rewrite the expansion to see ~

$$|v\rangle = \sum_m c_m |m\rangle = \sum_m |m\rangle c_m$$

$$= \sum_m |m\rangle\langle m|v\rangle = \left(\sum_m |m\rangle\langle m|\right)|v\rangle$$

Because $|v\rangle$ is arbitrary, it means that

$$\boxed{\sum_m |m\rangle\langle m| = \mathbb{1}}$$

check.

$$|1\rangle\langle 1| = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}$$

$$|6\rangle\langle 6| = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}-2 & 1\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}4 & -2\\-2 & 1\end{pmatrix}$$

$$\sum_m |m\rangle\langle m| = |1\rangle\langle 1| + |6\rangle\langle 6|$$

$$= \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix} + \frac{1}{5}\begin{pmatrix}4 & -2\\-2 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{1}$$

We can also construct the matrix M by its eigenvectors and eigenvalues ☺

$$\boxed{M = \sum_m m\,|m\rangle\langle m|}$$

Let's check.

$$|1\rangle\langle 1| = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$|6\rangle\langle 6| = \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\sum_m m\,|m\rangle\langle m| = 1\cdot|1\rangle\langle 1| + 6\cdot|6\rangle\langle 6|$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \frac{6}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 25 & -10 \\ -10 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$ — It is M!

Why can we construct the matrix M this way?

$$M|m\rangle = \sum_{m'} m'\,|m'\rangle\underbrace{\langle m'|m\rangle}_{\delta_{m'm}} = m\,|m\rangle$$

For an arbitrary vector $|v\rangle$,

$$|v\rangle = \sum_m |m\rangle\underbrace{\langle m|v\rangle}_{c_m}$$

$$= \sum_m c_m |m\rangle$$

When M acts on the vector $|v\rangle$

$$M|v\rangle = \sum_m M|m\rangle\langle m|v\rangle$$

$$= \sum_m m|m\rangle\langle m|v\rangle$$

$$= \left(\sum_m m|m\rangle\langle m|\right)|v\rangle$$

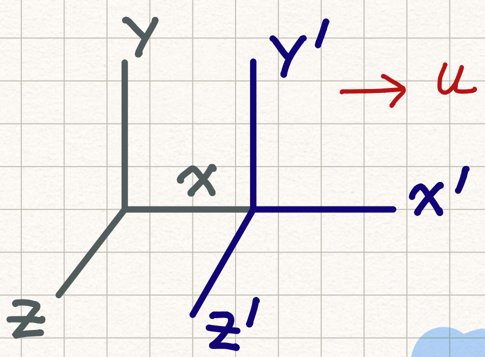
arbitrary

$$\rightarrow \quad M = \sum_m m|m\rangle\langle m|$$

YES!

Lorentz Transformation revisited ☺

Revisit the Lorentz transformation in 1+1 dim.



$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}}_{\text{matrix } L} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\cosh\alpha = \frac{1}{\sqrt{1-u^2/c^2}} \qquad \sinh\alpha = \frac{u/c}{\sqrt{1-u^2/c^2}}$$

Let's find the eigenvectors and eigenvalues of L ☺

$$\begin{vmatrix} \cosh\alpha - \lambda & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha - \lambda \end{vmatrix} = 0 \qquad (\lambda - \cosh\alpha)^2 - \sinh^2\alpha = 0$$

$$\rightarrow \quad \lambda_\pm = \cosh\alpha \pm \sinh\alpha$$

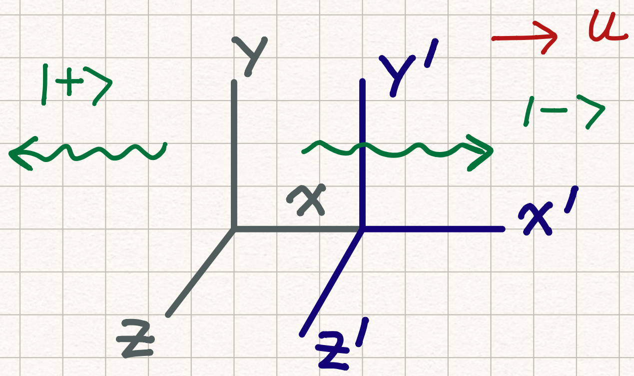
Now it is straightforward to find the eigenvectors

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_\pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\cancel{\cosh\alpha} \cdot a - \sinh\alpha \cdot b = (\cancel{\cosh\alpha} \pm \sinh\alpha)\, a$$

$$b = \mp a \quad \rightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the physical meaning of these eigenvectors and eigenvalues?



Light propagates along $\pm x$ axis

$$\boxed{x = \pm ct}$$

$$\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \right)$$

The Lorentz transformation also works for $(\omega, c\vec{k})$

$$\begin{pmatrix}\omega'\\ck'\end{pmatrix} = \begin{pmatrix}\cosh\alpha & -\sinh\alpha\\ -\sinh\alpha & \cosh\alpha\end{pmatrix}\begin{pmatrix}\omega\\ck\end{pmatrix}$$

$\omega = ck$ for light ☺

$$\rightarrow \quad \begin{pmatrix}\omega'\\\omega'\end{pmatrix} = \begin{pmatrix}\cosh\alpha & -\sinh\alpha\\ -\sinh\alpha & \cosh\alpha\end{pmatrix}\begin{pmatrix}\omega\\\omega\end{pmatrix}$$

$$\boxed{\omega' = (\cosh\alpha - \sinh\alpha)\,\omega}$$

$$\frac{\omega'}{\omega} = \frac{1}{\sqrt{1-u^2/c^2}} - \frac{u/c}{\sqrt{1-u^2/c^2}}$$

$$= \frac{(1-u/c)}{\sqrt{(1-\frac{u}{c})(1+\frac{u}{c})}}$$

Doppler effect!

$$= \sqrt{\frac{1-u/c}{1+u/c}} = \boxed{\sqrt{\frac{c-u}{c+u}}} < 1$$

Similarly, one can compute ω' for the light propagating along $-x$ axis : $(\omega, ck) = (\omega, -\omega)$.

$$\begin{pmatrix} \omega' \\ -\omega' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} \omega \\ -\omega \end{pmatrix}$$

$$\omega' = (\cosh\alpha + \sinh\alpha)\,\omega$$

$$\begin{aligned}
\frac{\omega'}{\omega} &= \frac{1}{\sqrt{1-u^2/c^2}} + \frac{u/c}{\sqrt{1-u^2/c^2}} \\
&= \frac{(1+u/c)}{\sqrt{(1-\frac{u}{c})(1+\frac{u}{c})}} \\
&= \sqrt{\frac{1+u/c}{1-u/c}} = \boxed{\sqrt{\frac{c+u}{c-u}}} > 1
\end{aligned}$$