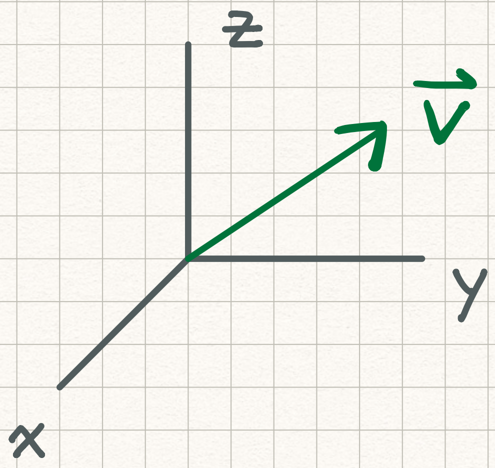
Vector Space



What is a vector, really?
While we are familiar with its presentation in Cartesian coordinates,

$$\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$$

we may not know its definition properly 😬

A set of objects $\vec{a}, \vec{b}, \vec{c}$ (vectors) forms a linear vector space $\mathbb{V}$ when satisfying:

① closed under additions ~

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

② closed under multiplication by scalars ~

$$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$$

$$(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$$

$$\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a}$$

③ existence of null vector $\vec{o}$

$$\vec{a} + \vec{o} = \vec{a}$$

④ unity scalar 1

$$1 \times \vec{a} = \vec{a}$$

⑤ existence of negative vector $-\vec{a}$

so that $\vec{a} + (-\vec{a}) = \vec{o}$.

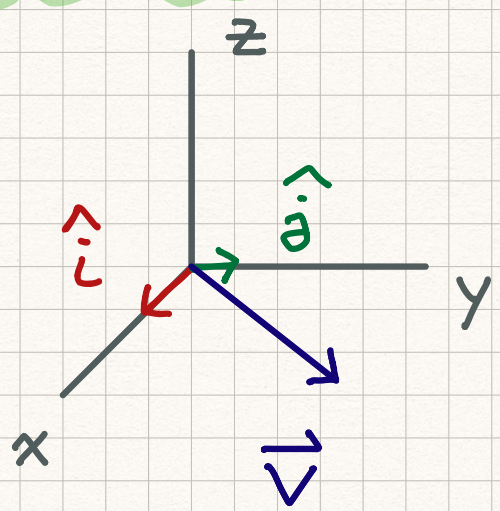
Linear Independence 😛

For N vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N$, one can construct the linear combination so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_N\vec{v}_N = \vec{0}$$

If the trivial soln $c_1 = c_2 = \cdots = c_N = 0$ is the ONLY soln, the set of vectors are **linearly independent**!

example

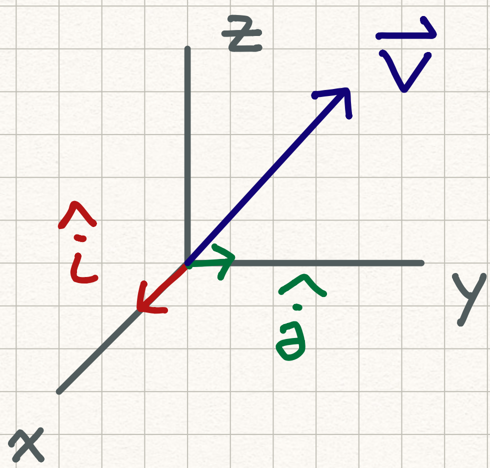


$$\vec{V} = V_x\hat{i} + V_y\hat{j}$$

$$\rightarrow \quad V_x\hat{i} + V_y\hat{j} - \vec{V} = 0$$

Thus, $\vec{V}, \hat{i}, \hat{j}$ are **linearly dependent** 😛

example $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$



① $c_1 \hat{i} + c_2 \hat{j} + c_3 \vec{V} = 0$

$$\rightarrow c_1 \underbrace{\hat{i} \cdot \hat{k}}_{=0} + c_2 \underbrace{\hat{j} \cdot \hat{k}}_{=0} + c_3 \underbrace{\vec{V} \cdot \hat{k}}_{V_z} = 0$$

Because $V_z \neq 0$, $c_3 = 0$

②

$c_1 \hat{i} + c_2 \hat{j} = 0$ applying similar trick

$$c_1 \hat{i} \cdot \hat{i} + c_2 \cancelto{0}{\hat{j} \cdot \hat{i}} = 0 \rightarrow c_1 = 0$$

$$c_1 \cancelto{0}{\hat{i} \cdot \hat{j}} + c_2 \hat{j} \cdot \hat{j} = 0 \rightarrow c_2 = 0$$

Thus, $\vec{V}$, $\hat{i}$, $\hat{j}$ are linearly independent!

The maximum number of linearly independent vectors in the vector space V is called the dimension of V.

In the above examples, the dimension of the vector space V is THREE

Inner Product revisited

The inner product has the following properties

1. $\langle a|b\rangle = \langle b|a\rangle^*$
2. $\langle a|\lambda b+\mu c\rangle = \lambda\langle a|b\rangle + \mu\langle a|c\rangle$

The above properties may sound a bit abstract. Thus, we often use an orthonormal basis to represent the vectors:

$$\langle \hat{e}_i|\hat{e}_j\rangle = \underbrace{\delta_{ij}}_{\text{kronecker delta symbol}} \qquad i,j = 1,2,\cdots,N$$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

An vector $|a\rangle$ can now be represented by $|\hat{e}_i\rangle$

$$|a\rangle = \sum_{i=1}^{N} a_i |\hat{e}_i\rangle$$

Because $|\hat{e}_i\rangle$ forms an orthonormal basis, the coefficients a_i can be computed easily

$$\langle \hat{e}_j|a\rangle = \sum_{i=1}^{N} a_i \underbrace{\langle \hat{e}_j|\hat{e}_i\rangle}_{\delta_{ji}} = a_j$$

One can derive the familiar formula for the inner product as below ~

$$|a\rangle = \sum_i a_i |\hat{e}_i\rangle \longrightarrow \langle a| = \sum_i a_i^* \langle \hat{e}_i|$$

$$|b\rangle = \sum_j b_j |\hat{e}_j\rangle$$

$$\langle a|b\rangle = \sum_i \sum_j a_i^* b_j \langle \hat{e}_i | \hat{e}_j \rangle$$

$$= \sum_i \sum_j a_i^* b_j \delta_{ij}$$

$$= \sum_i a_i^* b_i$$

— the familiar formula for inner product ☺

Linear Operator

A linear operator M maps a vector $|x\rangle$ to another vector $|y\rangle$: $|y\rangle = M|x\rangle$ with the following property ~

$$M\left(\lambda|a\rangle + \mu|b\rangle\right) = \lambda M|a\rangle + \mu M|b\rangle$$

Again, we can represent the linear operator M is an orthonormal basis

$$|y\rangle = A|x\rangle \qquad \begin{aligned} |x\rangle &= \sum_i x_j |\hat{e}_j\rangle \\ |y\rangle &= \sum_j y_j |\hat{e}_j\rangle \end{aligned}$$

$$A|x\rangle = \sum_j x_j A|\hat{e}_j\rangle = \sum_j y_j |\hat{e}_j\rangle$$

$$\Rightarrow \langle\hat{e}_i| \sum_j x_j A|\hat{e}_j\rangle = \langle\hat{e}_i|\sum_j y_j |\hat{e}_j\rangle$$

$$\sum_j x_j \underbrace{\langle\hat{e}_i|A|\hat{e}_j\rangle}_{\equiv A_{ij}} = \sum_j y_j \underbrace{\langle\hat{e}_i|\hat{e}_j\rangle}_{\delta_{ij}}$$

Finally, $\boxed{y_i = \sum_j A_{ij} x_j}$ — matrix algebra

4D Spacetime

The Lorentz transformation can be represented as a 4×4 matrix

$$\boxed{\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\frac{u}{c} & 0 & 0 \\ -\gamma\frac{u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}}$$

where $\gamma = \dfrac{1}{\sqrt{1-(u/c)^2}}$.

① Because $y' = y$, $z' = z$, we can focus on the 2×2 matrix.

② Introduce the hyperbolic parameter α

$$\cosh\alpha = \gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

$$\sinh\alpha = \frac{\gamma u}{c} = \frac{u/c}{\sqrt{1-(u/c)^2}}$$

$$\cosh^2\alpha - \sinh^2\alpha = \frac{1}{1-u^2/c^2} - \frac{u^2/c^2}{1-u^2/c^2}$$

$$= 1$$ yes ☺

Lorentz transformation simplified ~

$$\boxed{\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}}$$

For an object moving at const velocity v,

its trajector is $x = vt$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} ct \\ vt \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\alpha \cdot ct - \sinh\alpha \cdot vt \\ -\sinh\alpha \cdot ct + \cosh\alpha \cdot vt \end{pmatrix}$$

The velocity observed in the moving frame is

$$v' = \frac{x'}{t'} = \frac{\cosh\alpha \cdot v\cancel{t} - \sinh\alpha \cdot c\cancel{t}}{\cosh\alpha \cdot \cancel{t} - \sinh\alpha \cdot v\cancel{t}/c}$$

$$= \frac{v - \tanh\alpha \cdot c}{1 - \tanh\alpha \cdot v/c}$$

$\tanh\alpha = \frac{u}{c}$

$$\rightarrow \quad \boxed{v' = \frac{v - u}{1 - uv/c^2}}$$

(velocity addition in 4D spacetime !