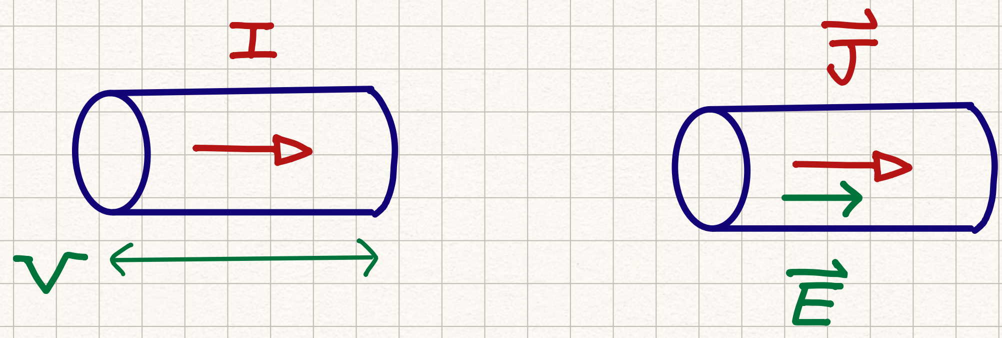
Convolution and Causality

Ohm's law describes the linear relation between I & V, or the professional version $\vec{J} = \sigma \vec{E}$.



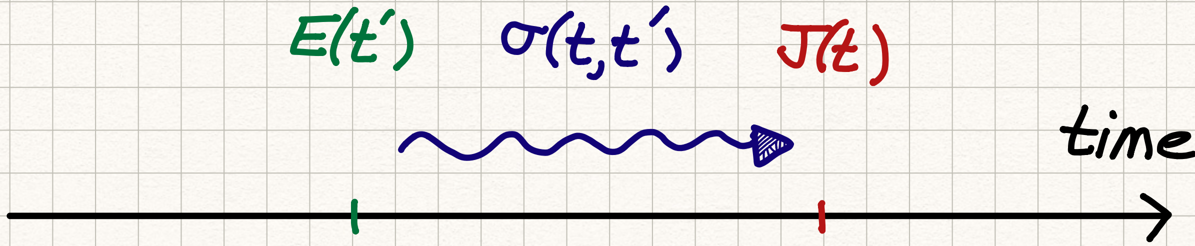
$$\underset{\text{response}}{I} = \frac{1}{R} \underset{\text{stimulus}}{V} \qquad \underset{\text{response}}{\vec{J}} = \sigma \underset{\text{stimulus}}{\vec{E}}$$

Skipping the vector notation, the Ohm's law is $J = \sigma E$. What happens when the stimulus is EM wave with non-zero ω? Can we just "generalize" the relation to

$$? \quad J(\omega) = \sigma(\omega)\, E(\omega) \quad ?$$

LTI (linear time-invariant) systems

Basics for linear response theory

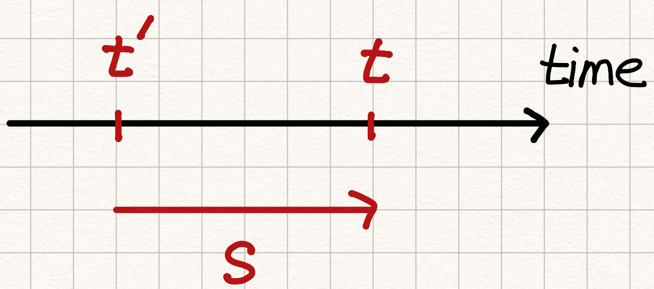


$$\boxed{J(t) = \int_{-\infty}^{+\infty} \sigma(t,t')\, E(t')\, dt'}$$

- translational symmetry in time

$$\sigma(t,t') = \sigma(t-t')$$

- causality $\sigma(t-t') = 0$ for $t-t' < 0$.



$$t - t' = s$$

$$J(t) = \int_{-\infty}^{+\infty} \sigma(t-t')\, E(t')\, dt'$$

$$= \int_{-\infty}^{+\infty} \sigma(s)\, E(t-s)\, ds$$

convolution ☺

Perform the Fourier transform.

$$\longrightarrow \quad J(\omega) = \sigma(\omega)\, E(\omega)$$

Convolution Theorem

Consider the following convolution integral:

$$\boxed{h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx}$$

$$\tilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz\, e^{-ikz} \int_{-\infty}^{+\infty} dx\, f(x)\, g(z-x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, f(x) \int_{-\infty}^{\infty} dz\, g(z-x)\, e^{-ikz}$$

change variable $u = z - x$ in the 2^{nd} integral

$$e^{-ikz} = e^{-ik(u+x)} = e^{-ikx} \cdot e^{-iku}$$

$$\tilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} \int_{-\infty}^{\infty} du\, g(u)\, e^{-iku}$$

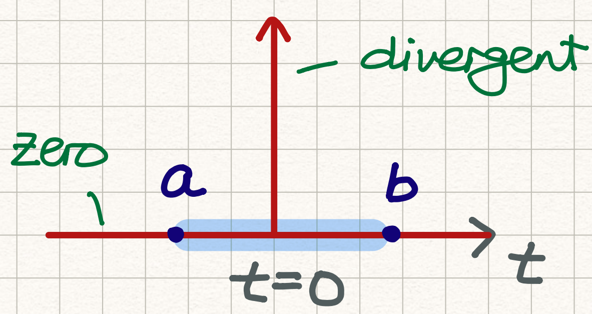
$$= \frac{1}{\cancel{\sqrt{2\pi}}}\, \cancel{\sqrt{2\pi}}\, \tilde{f}(k)\, \sqrt{2\pi}\, \tilde{g}(k) = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$

The convolution relation $h(z) = f * g(z)$ is simple after Fourier transform ~

$$\boxed{\tilde{h}(k) = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)}$$

depends on F.T. convention.

Dirac δ-function



Dirac introduced a rather singular fn $\delta(t)$ with the following properties:

- $\delta(t) = 0$, for $t \neq 0$ — $\delta(0)$ is NOT defined....

- $$\int_a^b dt\, f(t)\delta(t) = \begin{cases} f(0), & \text{if } 0 \in (a,b) \\ 0, & \text{if } 0 \notin (a,b) \end{cases}$$

One can choose $f(t) = 1$,

$$\int_a^b \delta(t)\, dt = 1 \quad \text{if } 0 \in (a,b)$$

That is to say, the Dirac δ-function integrates to unity.

Based on the integral form of definition, one can show the following identities for $\delta(t)$:

- $\delta(t) = \delta(-t)$ — even function
- $t\,\delta(t) = 0$ — annihilating δ-function
- $\delta(ct) = \frac{1}{|c|}\delta(t)$ — let's prove it

For $c > 0$, set $t' = ct$

$$\int_{-\infty}^{\infty} dt\, f(t)\, \delta(ct) = \int_{-\infty}^{+\infty} dt' \cdot \frac{1}{c}\, f(t'/c)\, \delta(t')$$

$$= \frac{1}{c} f(0) = \frac{1}{|c|} f(0) = \int_{-\infty}^{+\infty} dt\, f(t)\, \frac{1}{|c|}\delta(t)$$

$$\int_{-\infty}^{+\infty} f(t) \left[\delta(ct) - \frac{1}{|c|}\delta(t) \right] dt = 0$$

Because $f(t)$ is arbitrary, $\delta(ct) = \frac{1}{|c|}\delta(t)$.

For $c < 0$,

$$\int_{-\infty}^{+\infty} f(t)\, \delta(ct)\, dt = \int_{\infty}^{-\infty} f(t'/c)\, \delta(t')\, \frac{1}{c}\, dt'$$

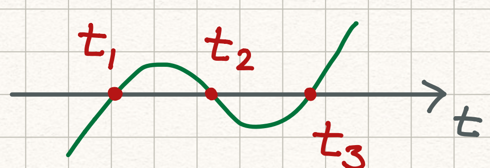
$$= -\frac{1}{c} \int_{-\infty}^{+\infty} f(t'/c)\, \delta(t')\, dt' = -\frac{1}{c} f(0) = \frac{1}{|c|} f(0).$$

Just follow the same steps $\longrightarrow$ $\delta(ct) = \frac{1}{|c|}\delta(t)$.

The above result can be generalized ...

$$\delta(h(t)) = \sum_n \frac{1}{|h'(t_n)|}\, \delta(t - t_n)$$

t_n are zeros of $h(t)$
i.e. $h(t_n) = 0$



$h(t) = (t - t_1)(t - t_2)(t - t_3)$

$$\delta(h(t)) = \frac{1}{|(t_1 - t_2)(t_1 - t_3)|}\, \delta(t - t_1) + \frac{1}{|(t_2 - t_1)(t_2 - t_3)|}\, \delta(t - t_2) + \frac{1}{|(t_3 - t_1)(t_3 - t_2)|}\, \delta(t - t_3)$$

Calculus for the Dirac δ-function

Starting from the basis property,

$$\int_{-\infty}^{+\infty} dt\, f(t)\,\delta(t) = f(0) \longrightarrow \quad f(t)\,\delta(t) = f(0)\,\delta(t)$$

Note that the identity holds true inside integral.

Does it make any sense to write $\delta'(t)$ at all?

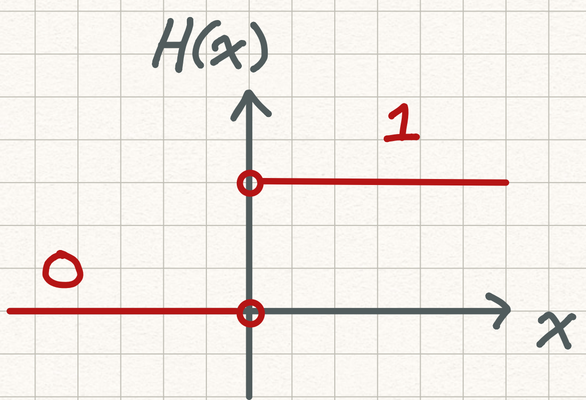
$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = \overset{0}{\cancel{f(t)\,\delta(t)}}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

$$= -f'(0)$$

$$\longrightarrow \quad f(t)\,\delta'(t) = -f'(0)\,\delta(t) = -f'(t)\,\delta(t)$$

On the other hand, one can also introduce the

Heaviside function $H(x)$ [unit step fn.]



$$\boxed{H(x) \equiv \int_{-\infty}^{x} \delta(s)\,ds}$$

- $\dfrac{dH}{dx} = \delta(x)$
- $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

What is $H(0)$ then?

$H(0) = \frac{1}{2}$?

Dirac δ-function in Fourier Transform

For every inversible integral transform, there exists a form of the Dirac δ-function ☺

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \tilde{f}(k) \, e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx' \, f(x') \, e^{-ikx'} \, e^{ikx}$$

$$= \int_{-\infty}^{+\infty} dx' \, f(x') \underbrace{\left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')} dk \right\}}_{\text{acting like a } \delta\text{-function!}}$$

$$\rightarrow \quad \boxed{\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ik(x-x')}}$$

Let's compute the Fourier transform of $\delta(x)$:

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; \delta(x) \, e^{-ikx} = \frac{1}{\sqrt{2\pi}} e^{-i0} = \frac{1}{\sqrt{2\pi}}$$

It is quite interesting that $\tilde{\delta}(k)$ is constant in the Fourier space ☺

Introduce a family of functions $\tilde{f}(k;\Lambda)$ in the Fourier space,

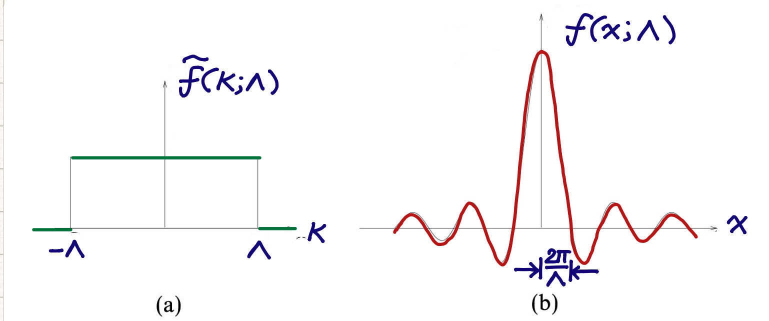


Figure 13.4 (a) A Fourier transform showing a rectangular distribution of frequencies between $\pm\Omega$; (b) the function of which it is the transform, which is proportional to $t^{-1}\sin\Omega t$.

For $-\Lambda < k < \Lambda$,

$$\tilde{f}(k;\Lambda) = \frac{1}{\sqrt{2\pi}}$$

(zero, otherwise)

$$f(x;\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\Lambda}^{\Lambda} dk\, e^{ikx} \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \cos(kx)\, dk$$

$$= \frac{1}{2\pi} \cdot \frac{1}{x} \left[\sin(\Lambda x) - \sin(-\Lambda x)\right]$$

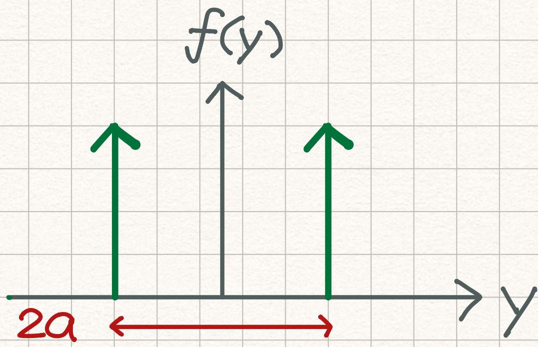
$$= \frac{\sin(\Lambda x)}{\pi x}$$

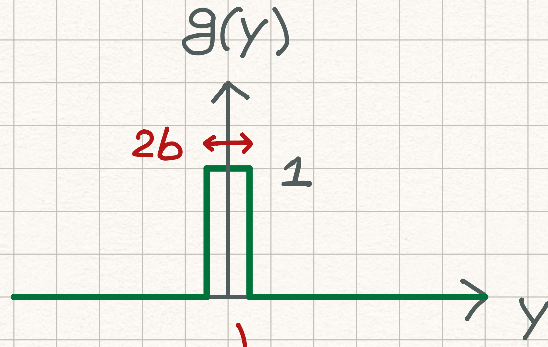
As Λ goes to infinity, $\lim_{\Lambda\to\infty} \tilde{f}(k;\Lambda) = \tilde{\delta}(k)$

$$\rightarrow \quad \boxed{\delta(x) = \lim_{\Lambda\to\infty} \frac{\sin(\Lambda x)}{\pi x}}$$

NOTE. It's easy to check that $\int_{-\infty}^{+\infty} f(x;\Lambda)\, dx = 1$

Doubt-Slit Interferences revisited





$$f(x) = \delta(x+a) + \delta(x-a)$$

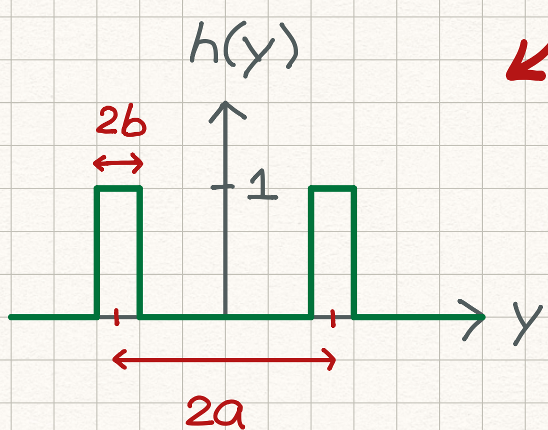
δ-fn denotes the locations of the two slits.

g(y) is the shape of a single slit.

$$h(y) = \int_{-\infty}^{\infty} f(s)\, g(y-s)\, ds$$

$$= \int_{-\infty}^{\infty} [\delta(s+a) + \delta(s-a)]\, g(y-s)\, ds$$

$$= g(y+a) + g(y-a)$$



From convolution theorem,

$$\tilde{h}(k) = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \cdot 2\cos(ka) \qquad \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \frac{2\sin(kb)}{k}$$

$$\longrightarrow \quad \boxed{\tilde{h}(k) = \frac{4\cos(ka)\sin(kb)}{\sqrt{2\pi}\, k}}$$