Complex Function

Let's start with the simplest case: polynomial.

$$z = x + iy \rightarrow f(z) = u + iv$$

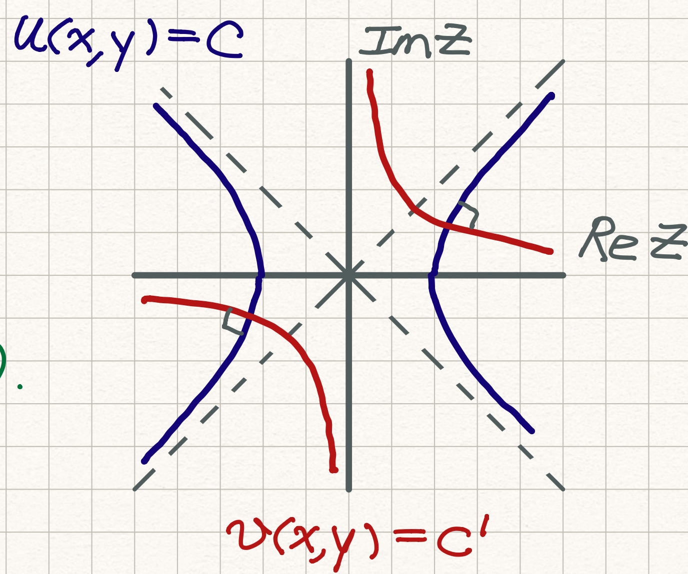
so, it's a mapping from (x, y) to (u, v)

example $f(z) = z^2$

$$f(z) = (x+iy)^2 = (x^2 - y^2) + i(2xy)$$

Thus, the real and imaginary parts are

$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$



① Plotting the equipotential contours for $u(x,y)$, $v(x,y)$.

They are orthogonal!

② u, v are harmonic functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \rightarrow \nabla^2 u = 0$$

similarly, $\nabla^2 v = 0$

trigonometric and hyperbolic functions

Making use of Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

one can generalize to complex variable $\theta \to z$ and obtain the expressions

$$\boxed{\begin{aligned} \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \end{aligned}}$$

The hyperbolic functions are defined as

$$\boxed{\begin{aligned} \cosh z &= \frac{1}{2}\left(e^{z} + e^{-z}\right) \\ \sinh z &= \frac{1}{2}\left(e^{z} - e^{-z}\right) \end{aligned}}$$

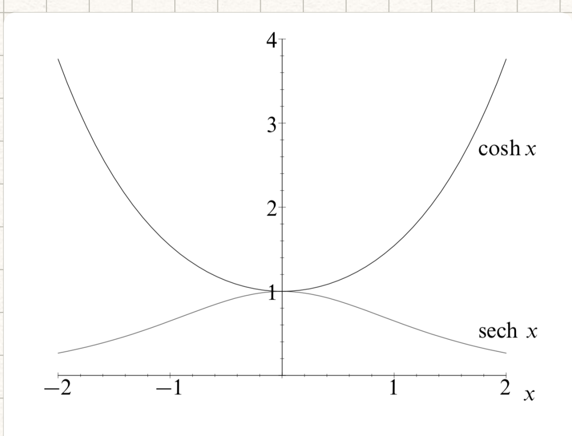


Figure 3.11 Graphs of $\cosh x$ and $\operatorname{sech} x$.

on the real axis $z = x$

$$\cosh x = \underbrace{\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}}_{\text{even function}}$$

Its reciprocal is $\operatorname{sech} x$

$$\operatorname{sech} x = (\cosh x)^{-1}$$

On the real axis $z = x$,

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

odd function

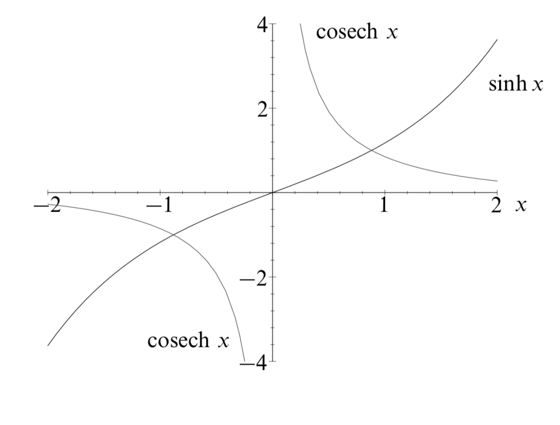


Figure 3.12 Graphs of sinh x and cosech$x$.

Its reciprocal is csch x

$$\operatorname{csch} x = (\sinh x)^{-1}$$

The "switch" function

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

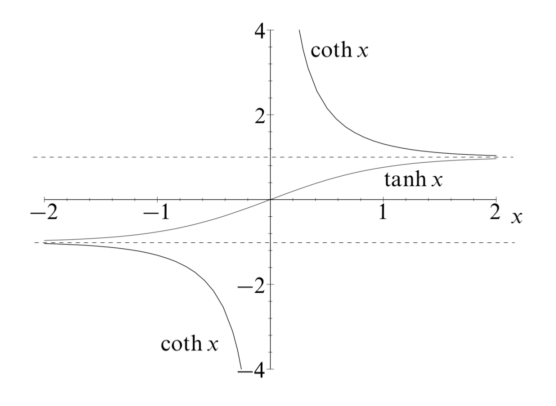


Figure 3.13 Graphs of tanh x and coth x.

It is clear from the figure that tanh x runs between −1 and +1

$$-1 \leq \tanh x \leq +1$$

Its reciprocal is $\coth x = (\tanh x)^{-1}$

From real to imaginary axis

Set $z = iy$ into trogonometric functions

$$\cos(iy) = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right)$$
$$= \frac{1}{2}\left(e^{-y} + e^{y}\right) = \cosh y$$

Similarly, one can plug $z = iy$ into $\sin z$.

$$\sin(iy) = \frac{1}{2i}\left(e^{-y} - e^{y}\right) = i \cdot \frac{1}{2}\left(e^{y} - e^{-y}\right)$$
$$= i \sinh y$$

$$\longrightarrow \boxed{\begin{aligned} \cos(iy) &= \cosh y \\ \sin(iy) &= i \sinh y \end{aligned}}$$

Trigonometric functions on the imaginary axis become hyperbolic functions on the real axis.

It is also interesting to compute $\cosh(iy)$ and $\sinh(iy)$. Or, even more general cases like $\cos(x+iy)$, $\sin(x+iy)$, $\cosh(x+iy)$, $\sinh(x+iy)$ .... Try it out.

Applications to D and $\int$

For instance, $f(x) = e^{3x}\cos 4x$ can be viewed as the real part of some function.

example. compute the derivative $\frac{df}{dx}$

$$g(x) = e^{3x}(\cos 4x + i\sin 4x) = e^{3x}e^{i4x}$$

$$= e^{(3+4i)x}$$

$$\frac{dg}{dx} = (3+4i)e^{(3+4i)x}$$

$$= e^{3x}(3+4i)(\cos 4x + i\sin 4x)$$

$$= e^{3x}(3\cos 4x - 4\sin 4x)$$

$$+ i\, e^{3x}(4\cos 4x + 3\sin 4x)$$

$$\frac{d}{dx}\left(e^{3x}\cos 4x\right) = Re\left(\frac{dg}{dx}\right)$$

$$= e^{3x}(3\cos 4x - 4\sin 4x)$$

Apply complex algebra to integration.

example $I = \int e^{ax} \cos bx \, dx$

$$g(x) = e^{ax}(\cos bx + i \sin bx) = e^{ax} e^{ibx}$$

$$= e^{(a+ib)x} \leftarrow \text{integrand} = \mathrm{Re}\, g(x)$$

$$W = \int g(x)\,dx = \frac{1}{a+ib} e^{(a+ib)x} + \text{Const.}$$

$$= \frac{(a-ib)}{(a+ib)(a-ib)} (\cos bx + i \sin bx) e^{ax} + \text{Const.}$$

$$= \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

$$+ i \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + \text{Const.}$$

$$I = \mathrm{Re}\, W$$

$$= \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + \text{Const.}$$

Complex logarithm

Introduce the logarithm function $\ln z$ on the complex plane and we run into some troubles.

$$z = re^{i\theta} = re^{i(\theta+2n\pi)}$$

$$\rightarrow \ln z = \ln\left[re^{i(\theta+2n\pi)}\right]$$

$$= \ln r + i(\theta + 2n\pi)$$

multiple valued ...

$\ln z$ is NOT a function!

↳ $\ln z$ is a multi-valued function.

example $\ln 1$

$$\ln 1 = \ln(e^{i2n\pi}) = i2n\pi = 0, \pm 2\pi i, \pm 4\pi i, \cdots$$

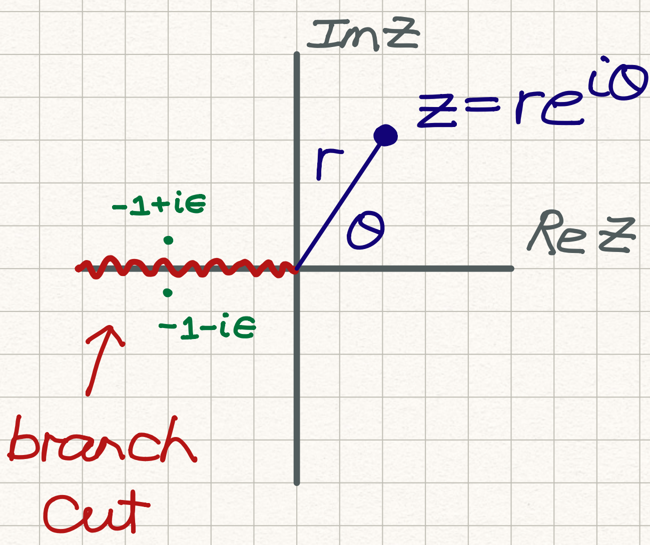
example $\ln i$

$$\ln i = \ln\left(e^{i(\frac{\pi}{2}+2n\pi)}\right) = i\left(\frac{\pi}{2}+2n\pi\right)$$

$$= \cdots, -\frac{3}{2}\pi i, \frac{1}{2}\pi i, \frac{5}{2}\pi i, \cdots$$

To make $\ln z$ a single-valued fn, one can limit the angular range

$$\ln z = \ln r + i\theta, \qquad -\pi < \theta \leq \pi$$



The price to pay is the "discontinuity" across the branch cut.

example $\ln(-1+i\epsilon)$, $\ln(-1-i\epsilon)$

$$\ln(-1+i\epsilon) = \ln\left(1e^{i(\pi-\delta)}\right) = i(\pi-\delta) = i\pi$$

$$\ln(-1-i\epsilon) = \ln\left(1e^{-i\pi+i\delta}\right) = -i\pi+i\delta = -i\pi$$

$$\ln(\underbrace{z^+}_{-1+i\epsilon}) - \ln(\underbrace{z^-}_{-1-i\epsilon}) = i\pi - (-i\pi) = 2\pi i \neq 0$$

discontinuous !!

complex powers

Logarithm function enables us to compute complex powers

$$(\text{complex number})^{\text{complex number}} = ?$$

example $2^i = ?$

$$2^i = e^{\operatorname{Ln} 2^i} = e^{i \operatorname{Ln} 2} = e^{i(\ln 2 + i2n\pi)}$$

$$= e^{i\ln 2} \cdot e^{-2n\pi}$$

$$= e^{-2n\pi}\left[\cos(\ln 2) + i\sin(\ln 2)\right]$$

$$\ldots, e^{-2\pi}, 1, e^{2\pi}, \ldots$$

So, 2^i is not $\cos(\ln 2) + i\sin(\ln 2)$. It actually has multiple values.

Example So, $i^2 = -1$, or not?

$$i^2 = e^{\operatorname{Ln}(i^2)} = e^{2\operatorname{Ln} i} = e^{2\operatorname{Ln}\left(1 e^{i\left(\frac{\pi}{2}+2n\pi\right)}\right)}$$

$$= e^{2\left(\ln 1 + i\left(\frac{\pi}{2}+2n\pi\right)\right)}$$

$$= e^{i(\pi+4n\pi)} = e^{i\pi} = -1$$

single-valued