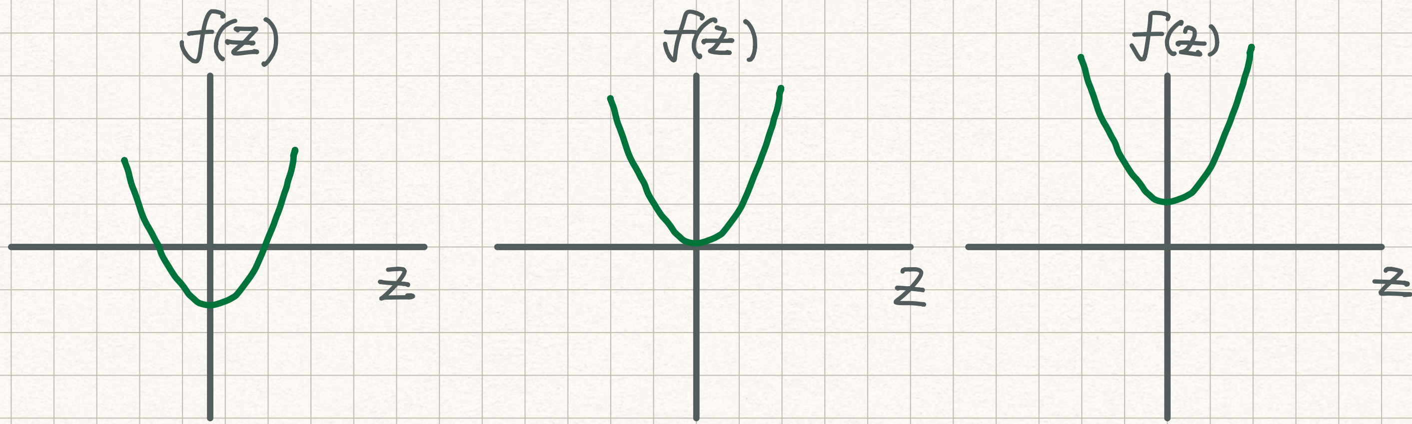
complex number

Why do we need complex numbers? ?!



$z^2 - 1 = 0$	$z^2 = 0$	$z^2 + 1 = 0$
two solutions	one solution	NO solution
$z = 1, -1$	$z = 0$	

But, one can also introduce an imaginary number $i \equiv \sqrt{-1}$ so that $i^2 = -1$

Thus, $z^2 + 1 = 0 \longrightarrow z^2 = -1$

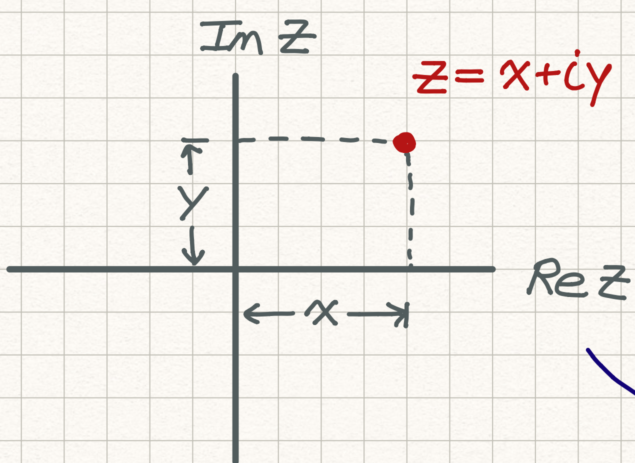
$$z = \pm\sqrt{-1} = \pm i$$

The imaginary number i extends the scope of algebra.

$i^2 = -1$, $i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

Complex plane From the multiplication pattern of the imaginary number i, one can construct a complex number $z = \underbrace{x}_{\text{real part}} + i\underbrace{y}_{\text{imaginary part}}$



The complex number can be represented on the 2D complex plane

The algebra of the complex number is simple.

$+$:
$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

$-$:
$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

$\times$:
$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

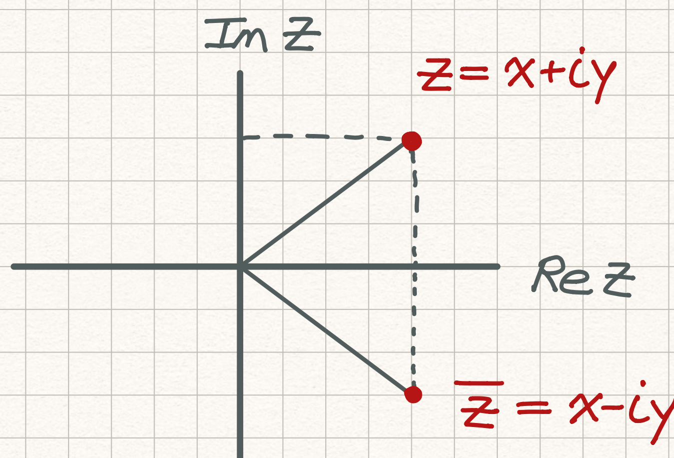
$\div$:
$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$

$$= \frac{(x_1x_2+y_1y_2)+i\,(y_1x_2-x_1y_2)}{x_2^2+y_2^2}$$

Finally,
$$\frac{z_1}{z_2} = \underbrace{\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}}_{\text{real part}} + i\,\underbrace{\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}}_{\text{imaginary part}}$$

The above calculation inspires a useful notion ~

complex conjugate $\bar{z} = x - iy$



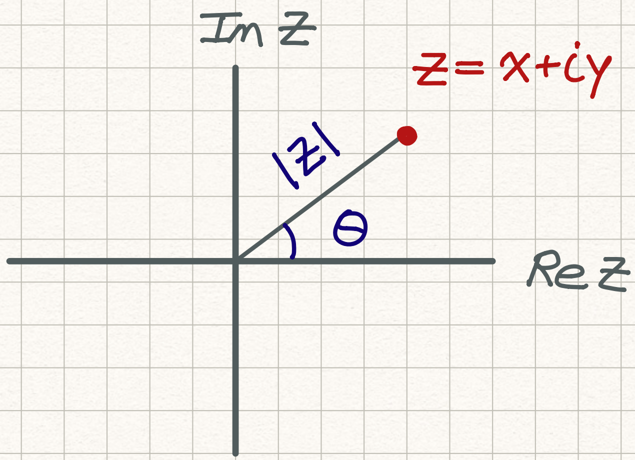
$$\bar{z}z = (x-iy)(x+iy)$$
$$= \underbrace{x^2+y^2}_{(\text{distance to origin})^2}$$

"complex conjugate" corresponds to mirror reflection with respect to the real axis ☺

Polar coordinates

modulus & argument

$|z|$ $\quad$ $\arg z$



It is easy to show that

$$|z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{\bar{z} z}$$

The argument θ is more subtle ...

$$\tan\theta = \frac{y}{x} \longrightarrow \arg z = \theta = \tan^{-1}\frac{y}{x}$$

For instance, $z_1 = x_1 + iy_1 = 1 + i$

$$\tan\theta_1 = \frac{y_1}{x_1} = 1$$

Because both x_1, y_1 are positive, $\arg z_1$ is in the 1^{st} quadrant.

On the other hand, $z_2 = x_2 + iy_2 = -1 - i$

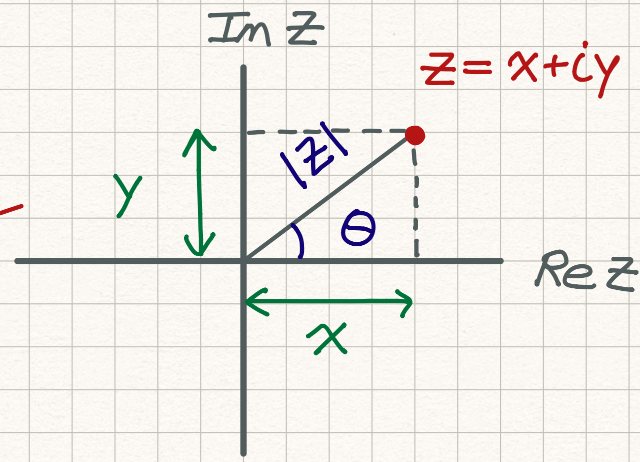
$$\tan\theta_2 = \frac{y_2}{x_2} = 1$$

equation looks the same ...

BUT! Because x_2, y_2 are both negative, $\arg z_2$ is in the 3^{rd} quadrant.

$$(x, y) \longleftrightarrow (|z|, \theta)$$

$$\boxed{\begin{aligned} x &= |z| \cos\theta \\ y &= |z| \sin\theta \end{aligned}}$$



Thus, the complex number z can be expressed as

$$z = x + iy = |z|\cos\theta + i\,|z|\sin\theta$$

$$= \underbrace{|z|}_{\text{modulus}} \underbrace{(\cos\theta + i\sin\theta)}_{\text{phase!}}$$

Euler's equation $e^{i\theta} = \cos\theta + i\sin\theta$

The exponential fn e^z can be expanded as

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

For $z = i\theta$ (θ is real), it can be written as

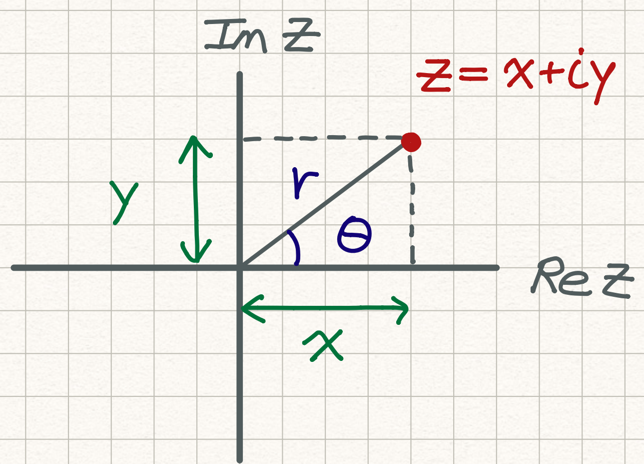
$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad \leftarrow \cos\theta$$

$$+ i\left(\theta - \frac{1}{3!}\theta^3 + \cdots\right) \quad \leftarrow \sin\theta$$

$$\Rightarrow \quad e^{i\theta} = \cos\theta + i\sin\theta$$ — Euler's equation

This relation is surprisingly beautiful, connecting the exponential fn and sinusoidal fns together!



Two representations in the complex plane

$$z = x + iy$$
$$= re^{i\theta}$$

de Moivre's theorem

Making use of the relation $(e^{i\theta})^n = e^{in\theta}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

The above result is called de Moivre's thm.

It is useful in finding trigonometric identities

example $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\cos^3\theta - 3\cos\theta\sin^2\theta$$

$$+ i\,(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$1 - \cos^2\theta$

$$\rightarrow \quad \cos 3\theta = \cos^3\theta - 3\cos\theta\,\underline{\sin^2\theta}$$

$$= 4\cos^3\theta - 3\cos\theta \quad \#$$

example $\quad \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

set $z = e^{i\theta}$, $\quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$$

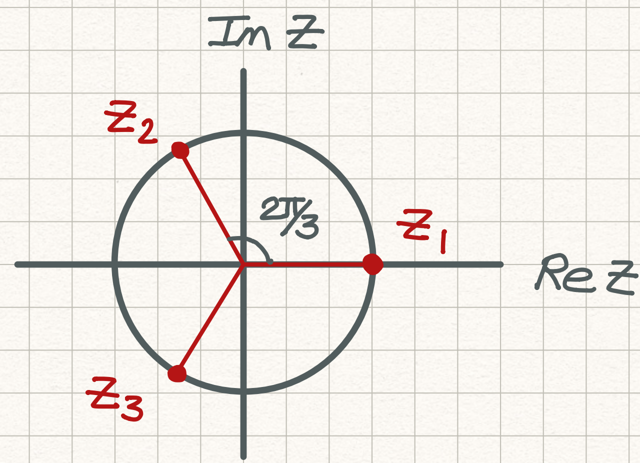
Thus, $\cos^3\theta = \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3$

$$= \frac{1}{8}\left(z^3 + 3z^2\cdot\frac{1}{z} + 3z\cdot\frac{1}{z^2} + \frac{1}{z^3}\right)$$

$$= \frac{1}{8}\left(z^3+\frac{1}{z^3}\right) + \frac{3}{8}\left(z+\frac{1}{z}\right)$$

$$= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta \quad \#$$

example $z^3 = 1$



$z = re^{i\theta}$

$$r^3 e^{i3\theta} = 1 = e^{i2n\pi}$$

Thus, the modulus $r=1$.

$$e^{i\theta} = e^{i\frac{2n\pi}{3}} = 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$$ three solutions

The solutions are 3 phases : $z_n = e^{i2n\pi/3}$

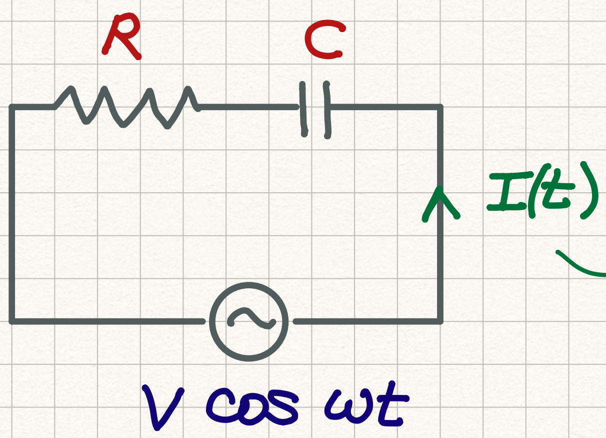
It's easy to see from the complex plane that

$z_1+z_2+z_3 = 0$ — $1+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$+\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

$=\left(1-\frac{1}{2}-\frac{1}{2}\right)+i\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)=0$

electrical impedance Z — physics application



One can try to solve the current $I(t)$ by ordinary differential equation.

OR, one can introduce the complex impedance Z.

$$Z = Z_R + Z_C = R + \frac{1}{i\omega C}$$

Treating both I, V as complex numbers, it can be shown that Ohm's law still holds.

$$V = IZ \longrightarrow I(t) = \frac{V(t)}{Z}$$

Let's see how this works

$$V(t) = Ve^{i\omega t} = \underbrace{V\cos\omega t}_{\text{"real" voltage}} + iV\sin\omega t$$

The RC impedance is $Z = |Z|e^{i\phi}$

$$I(t) = \frac{V(t)}{Z} = \frac{Ve^{i\omega t}}{|Z|e^{i\phi}} = \frac{V}{|Z|}e^{i(\omega t - \phi)}$$

$$I(t) = \underbrace{\frac{V}{|Z|}\cos(\omega t - \phi)}_{\text{"real" current}} + i\frac{V}{|Z|}\sin(\omega t - \phi)$$

Now, one only need to compute the modulus and the phase.

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$ — modulus.

$$\tan\phi = -\frac{1}{\omega RC}$$ — phase

Because $Z = R - i\frac{1}{\omega C}$, the phase ϕ is in the 4th quadrant. Some people prefer defining $-\phi$ as the phase.

Q: In the high-frequency limit ($\omega \to \infty$), the impedance $Z_c = \frac{1}{i\omega C} \to 0$. Why?

In the $\omega \to \infty$ limit, the current oscillation is extremely fast and charges cannot build up in the capacitor. No charge, no voltage, no effect!

Is "z" a 2D vector?

From the addition/substraction algebra

$$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$$

It looks like 2D vectors.

Introduce complex conjugate $\bar{z}\equiv x-iy$

$$\bar{z}_1 z_2=(x_1-iy_1)(x_2+iy_2)$$
$$=(x_1x_2+y_1y_2)+i(x_1y_2-y_1x_2)$$

Geometric meaning ~ $\vec{r}_1=(x_1,y_1)$, $\vec{r}_2=(x_2,y_2)$

inner product $\vec{r}_1\cdot\vec{r}_2=x_1x_2+y_1y_2$

outer product $\vec{r}_1\times\vec{r}_2=(x_1y_2-y_1x_2)\hat{k}$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_1 & y_1 & 0\\ x_2 & y_2 & 0\end{vmatrix}$$

$$\boxed{\begin{aligned}\mathrm{Re}\,(\bar{z}_1z_2)&=\vec{r}_1\cdot\vec{r}_2\\ \mathrm{Im}\,(\bar{z}_1z_2)&=(\vec{r}_1\times\vec{r}_2)_z\end{aligned}}$$

The complex number is a 2D vector. Furthermore, its algebra "integrates" both inner and outer products!