Matrix

Linear operators can be represented by matrices.

$$A|x\rangle = |y\rangle \rightarrow \sum_j A_{ij} x_j = y_i$$

where the matrix presentation of A is

$$\boxed{A_{ij} = \langle \hat{e}_i | A | \hat{e}_j \rangle}$$

Note that the vectors can be presented as bras & kets.

$$|x\rangle = \sum_i x_i |\hat{e}_i\rangle$$ ← ket column vector

$$\langle x| = \sum_i \langle \hat{e}_i | x_i^*$$ ← bra row vector

The inner product is formed by bra & ket!

$$\underbrace{\langle x|}_{\text{bra}}\underbrace{|y\rangle}_{\text{ket}} = \sum_i x_i^* y_i$$

It's complex ☺

bracket (inner product)

It's also easy to see that

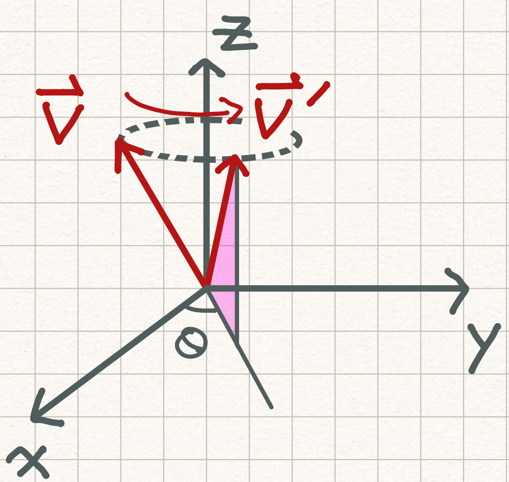
$$\langle y|x\rangle = \sum_i y_i^* x_i = \langle x|y\rangle^*$$

matrix algebra

The matrix algebra can be derived from the properties of linear operators.

$$\boxed{\begin{aligned} (A+B)_{ij} &= A_{ij} + B_{ij} \\ (\lambda A)_{ij} &= \lambda A_{ij} \\ (AB)_{ij} &= \sum_k A_{ik} B_{kj} \end{aligned}}$$

example: rotations in 3D



Rotate along the z axis by angle θ:

$$R_z(\theta)\,\vec{V} = \vec{V}'$$

The components satisfy the linear relations:

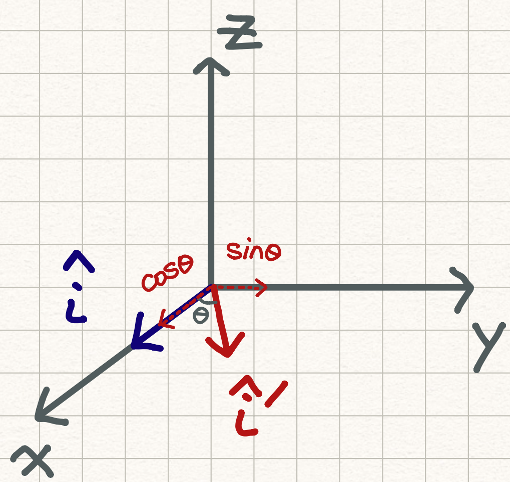
$$V_x' = V_x \cos\theta - V_y \sin\theta$$
$$V_y' = V_x \sin\theta + V_y \cos\theta$$
$$V_z' = V_z$$

V_x, V_y mix up

One can represent $R_z(\theta)$ in matrix form,

$$\boxed{\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}}$$

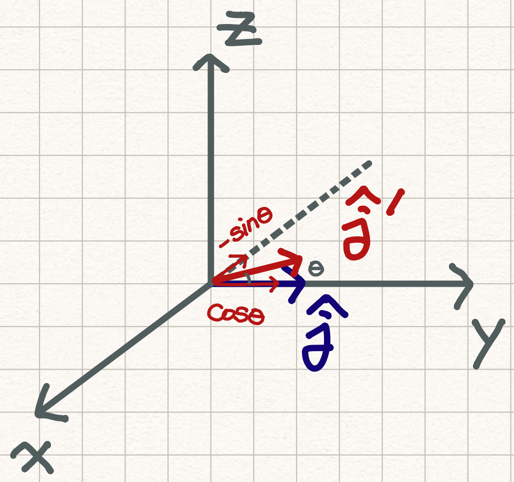
Let us construct the matrix by rotating the basis vectors $\hat{i}, \hat{j}, \hat{k}$



(1) Apply $R_z(\theta)$ on $\hat{i}$:

$$\hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

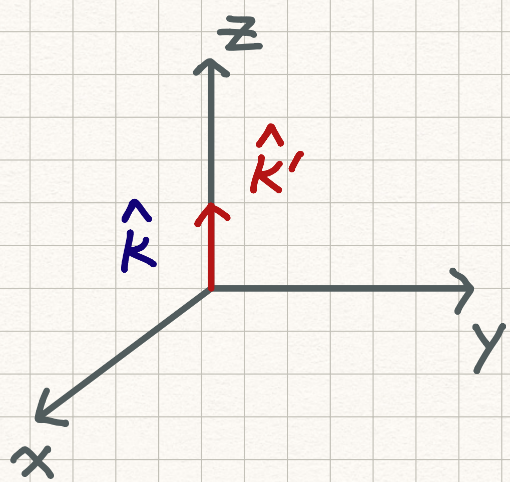
$$\begin{pmatrix} i_x' \\ i_y' \\ i_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & & \\ \sin\theta & & \\ 0 & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



(2) Apply $R_z(\theta)$ on $\hat{j}$:

$$\hat{j}' = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

$$\begin{pmatrix} j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} & -\sin\theta & \\ & \cos\theta & \\ & 0 & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



(3) Apply $R_z(\theta)$ on $\hat{k}$:

$$\hat{k}' = \hat{k}$$

$$\begin{pmatrix} k'_x \\ k'_y \\ k'_z \end{pmatrix} = \begin{pmatrix} & & 0 \\ & & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Commutator $[A, B] \equiv AB - BA$

Now that we learn how to construct $R_z(\theta)$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can also write down $R_x(\theta)$, $R_y(\theta)$:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

For simplicity, set all rotation angles $\theta = \frac{\pi}{2}$ in R_x, R_y, R_z operators

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \bar{1} \\ 0 & 1 & 0 \end{pmatrix} \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \bar{1} & 0 & 0 \end{pmatrix} \quad R_z = \begin{pmatrix} 0 & \bar{1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let's work out the commutator $[R_x, R_y] = R_x R_y - R_y R_x$

$$R_x R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \bar{1} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \bar{1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_y R_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \bar{1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \bar{1} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \bar{1} \\ \bar{1} & 0 & 0 \end{pmatrix}$$

It's clear that $R_x R_y \neq R_y R_x$ so $[R_x, R_y] \neq 0$.

One can check that this is a generic feature of rotations in 3D!

Transpose of a Matrix

By interchanging the rows and columns of A, its transpose A^T is defined as

$$(A^T)_{ij} = A_{ji}$$

example transpose of rotation matrices ☺

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R^T)_{ij} = R_{ji}$$

$$R_z^T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's muliply them together ☺

$$R_z^T(\theta) R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

kronecker delta δ_{ij}

Orthogonal Matrix

It is quite interesting that $R_z^T R_z = \mathbb{1}$. For matrices with the property, they are called orthogonal matrices!

The name is puzzling, isn't it? As explained before, the matrix $R_z(\theta)$ can be viewed as 3 rotated basis vectors.

$$\langle \hat{e}_i | \hat{e}_j \rangle = \delta_{ij} \longrightarrow \langle \hat{e}_i' | \hat{e}_j' \rangle = \delta_{ij}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It just means that $R_z^T \cdot R_z = \mathbb{1}$.

Hermitian Conjugate

Generalize the idea of "transpose" to the complex matrices $\longrightarrow$ Hermitian Conjugate !

$$(A^{+})_{ij} = A^{*}_{ji}$$

If the matrix is real, Hermitian Conjugate is the same as transpose ☺

example $\quad (AB)^{+} = B^{+}A^{+}$

$$[(AB)^{+}]_{ij} = (AB)^{*}_{ji} = \sum_{k} A^{*}_{jk} B^{*}_{ki}$$

$$= \sum_{k} (A^{+})_{kj} (B^{+})_{ik}$$

It's OK to change the order !

$$= \sum_{k} (B^{+})_{ik} (A^{+})_{kj}$$

$$= (B^{+}A^{+})_{ij}$$

It shall be clear that $(AB)^{T} = B^{T}A^{T}$ also holds for the transpose ☺