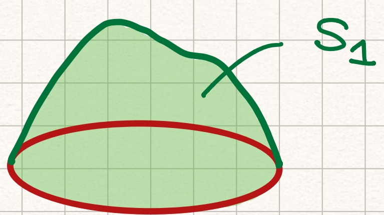
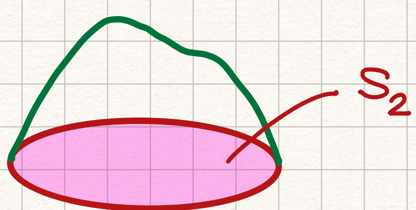
Stokes' Theorem

Recall the definition of curl,

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\partial_y V_z - \partial_z V_y,\ \partial_z V_x - \partial_x V_z,\ \partial_x V_y - \partial_y V_x\right)$$

Stokes' theorem $\boxed{\oint_{\partial S} \vec{V}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a}}$

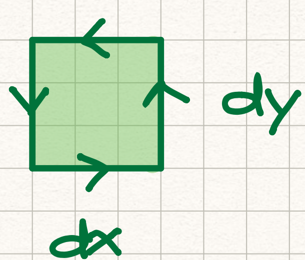




$S_1 \neq S_2$ but $\partial S_1 = \partial S_2$, Stokes' theorem still works

Stokes' theorem doesn't work for 1-side surface (like Möbius strip)

Let's prove the theorem by Ms Loop on the x-y plane :



circulation $= \oint_{\partial S} \vec{V}\cdot d\vec{r}$

$= [V_x(y) - V_x(y+dy)]\,dx$

$+ [V_y(x+dx) - V_y(x)]\,dy$

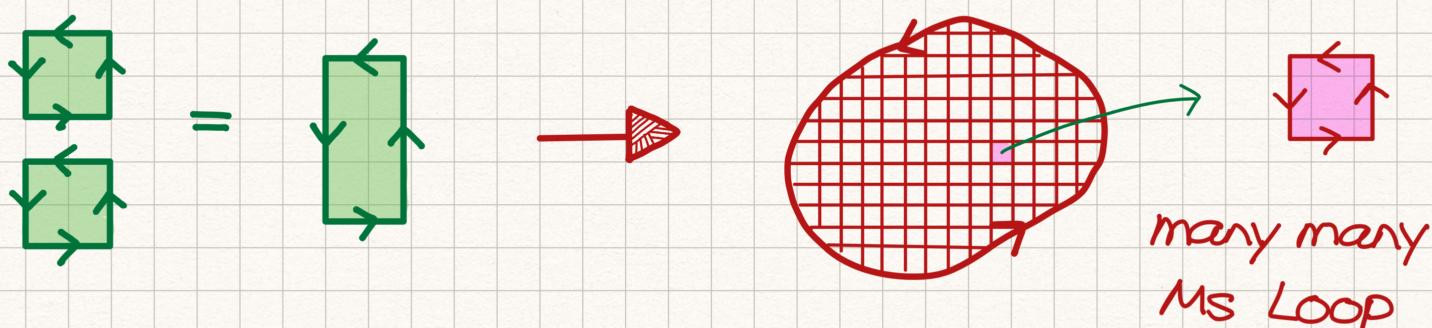
$$\oint_{\partial S} \vec{V}\cdot d\vec{r} = -\frac{\partial V_x}{\partial y}\,dy\,dx + \frac{\partial V_y}{\partial x}\,dx\,dy$$

$$= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy$$

$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a} = (\vec{\nabla}\times\vec{V})\cdot\hat{k}\,dx\,dy$$

$$= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy$$

Thus, the circulation can also be computed by the surface integral of the curl.



So, the Stokes' theorem works for arbitrary closed contours on the x-y plane:

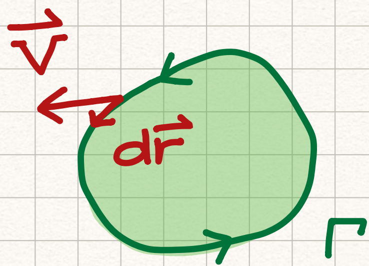
$$\boxed{\oint_{\partial S} \vec{V}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a}}$$

The above results can be generalized to the 3D case (not a trivial task).

Conservative Field

Consider an irrotational vector field $\vec{V}$:

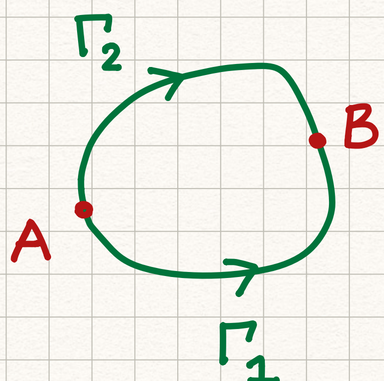
$$\vec{\nabla} \times \vec{V} = 0 \quad \Rightarrow \quad \int_S \vec{\nabla} \times \vec{V} \, d\vec{a} = 0$$



According to the Stokes' thm,

$$\boxed{\oint_{\partial S} \vec{V} \cdot d\vec{r} = 0}$$

It's quite interesting that circulation along any closed contour is ZERO!

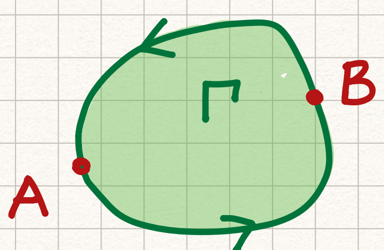


Consider two line integrals

$$I_1 = \int_{\Gamma_1} \vec{V} \cdot d\vec{r}$$

$$I_2 = \int_{\Gamma_2} \vec{V} \cdot d\vec{r}$$

Reversing Γ_2 to combine Γ_1 into a closed contour Γ.



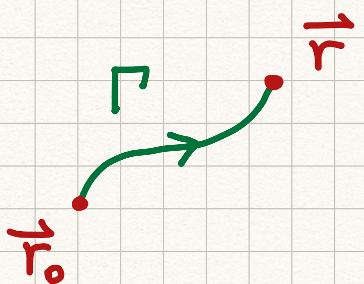
$$\underbrace{\oint_{\Gamma} \vec{V} \cdot d\vec{r}}_{\text{zero}} = \int_{\Gamma_1} \vec{V} \cdot d\vec{r} - \int_{\Gamma_2} \vec{V} \cdot d\vec{r}$$

$$\Rightarrow \quad \int_{\Gamma_1} \vec{V} \cdot d\vec{r} = \int_{\Gamma_2} \vec{V} \cdot d\vec{r} = \phi(\vec{r}_B) - \phi(\vec{r}_A)$$

The important conclusion here is ~

The notion of potential $\phi(\vec{r})$ works for an irrotational vector field $\vec{V}(\vec{r})$.

We may go further to derive the relation between ϕ and $\vec{V}$



Because $\vec{\nabla}\times\vec{V}=0$, the potential can be defined,

$$\phi(\vec{r}) = \phi(\vec{r}_0) + \int_\Gamma \vec{V}\cdot d\vec{r}$$

But, from the relation $d\phi = \vec{\nabla}\phi\cdot d\vec{r}$

$$\Delta\phi = \phi(\vec{r}) - \phi(\vec{r}_0) = \int_\Gamma \vec{\nabla}\phi\cdot d\vec{r}$$

$$\Rightarrow \quad \phi(\vec{r}) = \phi(\vec{r}_0) + \int_\Gamma \vec{\nabla}\phi\cdot d\vec{r}$$

Thus, $\int_\Gamma [\vec{V}-\vec{\nabla}\phi]\cdot d\vec{r} = 0$ — for arbitrary contour Γ !

It means that $\boxed{\vec{V} = \vec{\nabla}\phi}$

(sometimes with a minus sign

Example $\vec{E}$ field of a point charge Q

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$(\vec{\nabla}\times\vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{Q}{4\pi\epsilon_0}\Big[y\cdot\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x$$

$$- x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2y\Big] = 0$$

One can also compute the other components,

$\vec{\nabla}\times\vec{E} = 0 \longrightarrow$ potential $\phi(\vec{r})$ exists !

One can construct the Coulomb potential

$$\phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

note that $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r}$

$$\nabla\phi = -\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = -\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} = -\vec{E}$$

$\longrightarrow$ $\boxed{\vec{E} = -\vec{\nabla}\phi}$ — This relation holds true for electrostatics.

NOTE. From Maxwell equation (Faraday's law)

irrotational !

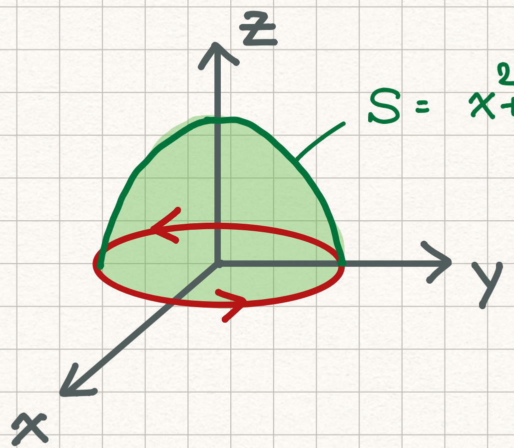
$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \xrightarrow{\text{static}} \vec{\nabla}\times\vec{E} = 0$$

Thus, $\boxed{\vec{E} = -\vec{\nabla}\phi}$

In the presence of ,

$\vec{E} \neq -\vec{\nabla}\phi$ ... careful

example Circulation of the field



$S: x^2+y^2+z^2=a^2$
$z \geq 0$

Given a vector field

$$\vec{V} = (4y, x, 2z),$$

want to compute

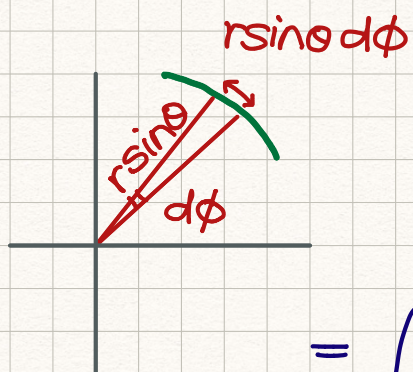
$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a} = ?$$

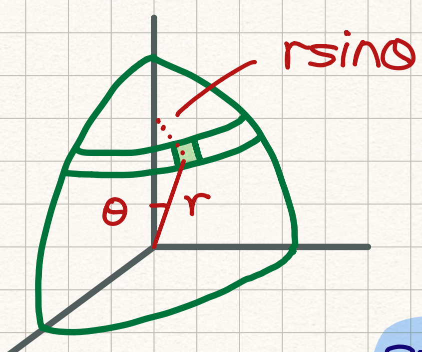
There are 3 ways to go

Method I. Directly evaluate the surface integral.

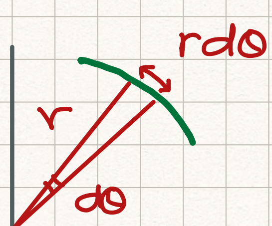
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & x & 2z \end{vmatrix} = (0,0,-3) = -3\hat{k}$$

TOP view





SIDE view



$$d\vec{a} = r^2\sin\theta\, d\theta\, d\phi\, \hat{n}$$

$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a}$$

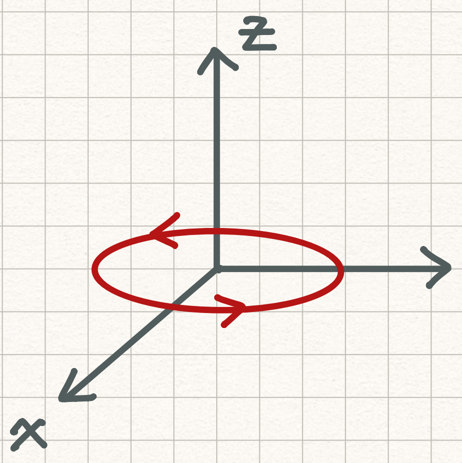
$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, -3\,\overbrace{\hat{k}\cdot\hat{n}}^{\cos\theta}\, r^2\sin\theta$$

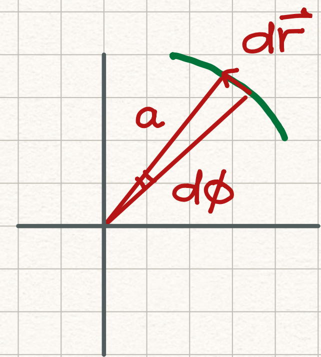
$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, -3a^2\sin\theta\cos\theta$$

$$= -\frac{3}{\cancel{2}}a^2\cdot\cancel{2}\pi \underbrace{\int_0^{\pi/2} d\theta\, \sin 2\theta}_{=1}$$

$$= -3\pi a^2$$

Method II. Make use of the Stokes' theorem.





$$d\vec{r} = dr\,\hat{e}_\phi = a\,d\phi\left(-\frac{y}{a}, \frac{x}{a}, 0\right)$$

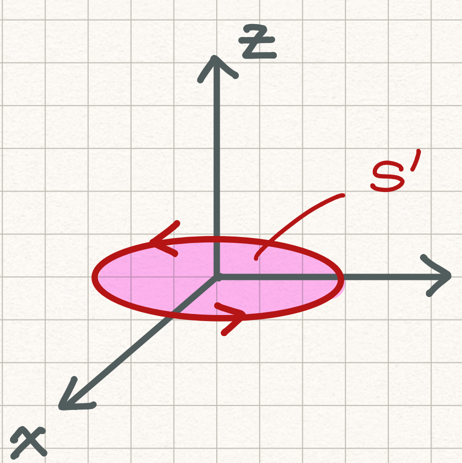
$$\begin{aligned}\vec{V}\cdot d\vec{r} &= (4y, x, 2z)\cdot(-y, x, 0)\,d\phi \\ &= (-4y^2+x^2)\,d\phi \\ &= (-4\sin^2\phi+\cos^2\phi)\,a^2\,d\phi \\ &= (1-5\sin^2\phi)\,a^2\,d\phi\end{aligned}$$

$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a} = \int_{\partial S}\vec{V}\cdot d\vec{r} = \int_0^{2\pi} a^2(1-5\sin^2\phi)\,d\phi$$

$$= a^2\cdot\left(1-\frac{5}{2}\right)\cdot 2\pi = -3\pi a^2$$

the same result

Method III. Choose another surface S'

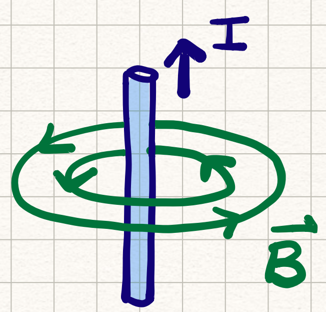


$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a} = \int_{S'} \vec{\nabla}\times\vec{V}\cdot d\vec{a}$$

$$= \int_{S'} -3\hat{k}\cdot\hat{n}\,da$$

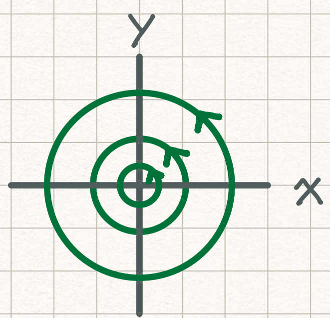
$$= -3\pi a^2$$ — remarkable!

Vortex in the Field



The magnetic field around an infinitely long wire is

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\left(\frac{-y}{\rho}, \frac{x}{\rho}, 0\right)}$$



One can compute the curl:

$\vec{\nabla} \times \vec{B} = 0$ except @ the origin.

Introduce the magnetic potential.

$$V_B = \frac{\mu_0 I}{2\pi}\phi = \frac{\mu_0 I}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\vec{B} = \vec{\nabla} V_B = \left(\frac{\partial V_B}{\partial x}, \frac{\partial V_B}{\partial y}, \frac{\partial V_B}{\partial z}\right)$$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{1}{1+(y/x)^2}\cdot -\frac{y}{x^2}, \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}, 0\right)$$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

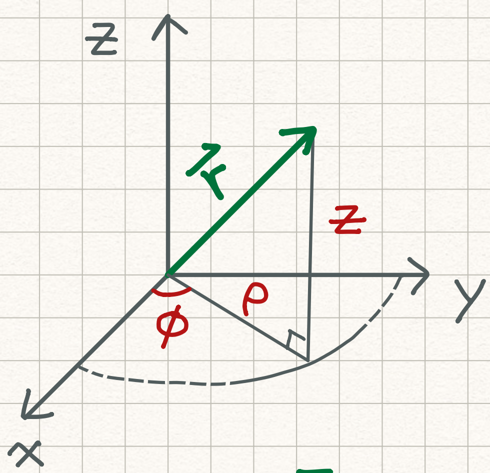
Correct!

Because the $\vec{B}$ field can be expressed as $\vec{\nabla} V_B$,

$$\oint_\Gamma \vec{B}\cdot d\vec{r} = \oint_\Gamma \vec{\nabla} V_B \cdot d\vec{r} = 0 \,?$$

From the Ampere's law, $\oint_\Gamma \vec{B}\cdot d\vec{r} = \mu_0 I \neq 0$

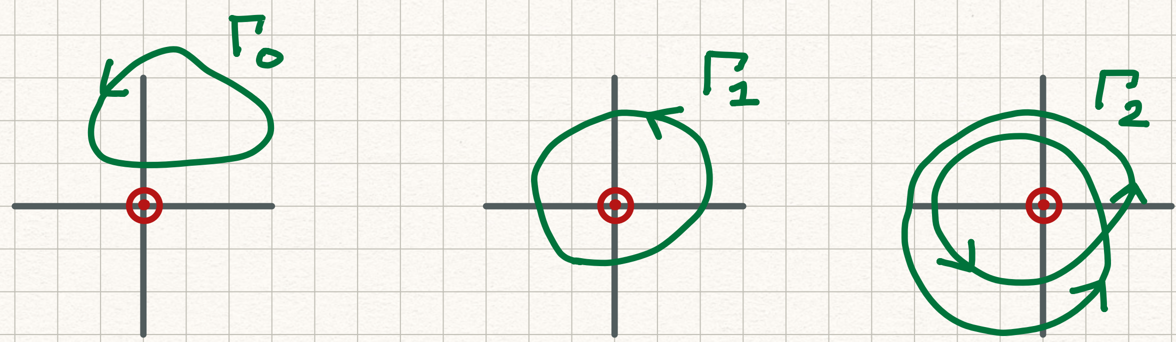
In cylindrical coordinates (ρ, ϕ, z)



$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{\rho}\,\hat{e}_\phi$$

and the potential is

$$V_B = \frac{\mu_0 I}{2\pi}\cdot\phi$$



$\Delta\phi = 0$ $\Delta\phi = 2\pi$ $\Delta\phi = 4\pi$

$$\oint_\Gamma \vec{B}\cdot d\vec{r} = \mu_0 I\cdot W$$

W is the winding number of Γ.

NOTE. If one computes the curl carefully, the answer is not zero,

$$\vec{\nabla}\times\vec{B} = \mu_0 I\,\delta(x)\delta(y)$$

So, $\oint_{\partial S} \vec{B}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{a} = \mu_0 I$