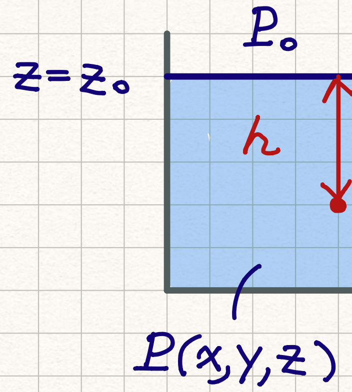
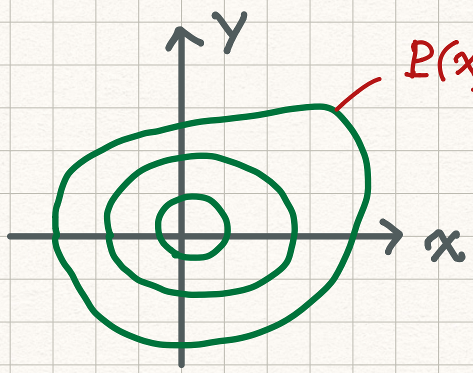
Vector Calculus

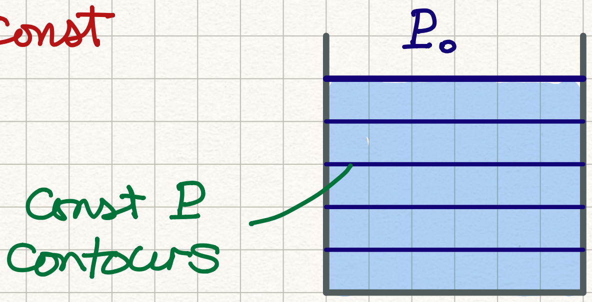
The concept of field in natural sciences is of crucial importance



$$\underbrace{P(x,y,z)}_{\text{scalar field}} = \overbrace{P_0}^{\text{pressure on the surface}} + \rho g \underbrace{(z_0 - z)}_{\text{depth beneath the surface}}$$

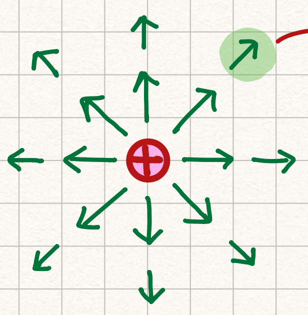
How will you visualize a scalar field?

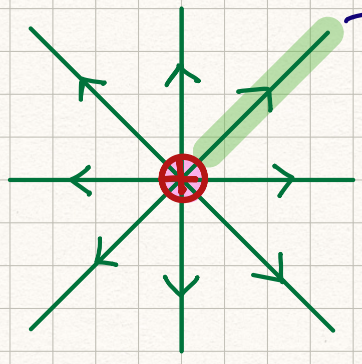




Now we can proceed to vector fields.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad r = \sqrt{x^2+y^2+z^2} \quad \text{and} \quad \hat{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$





works only $\vec{\nabla}\cdot\vec{E} = 0$

Vector Operators

How can one describe the changes of the fields?

By their derivative ...

Gradient $$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Divergence $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Curl $$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Laplacian $$\nabla^2\phi = \nabla\cdot\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

∇^2 can also act on vector fields

$\nabla^2\vec{V} = ?$ work it out!

Note

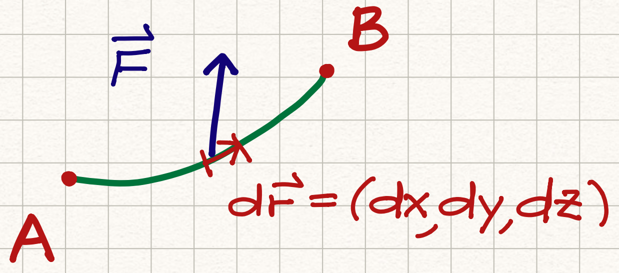
$\vec{\nabla}\phi$ scalar → vector

$\vec{\nabla}\cdot\vec{V}$ vector → scalar

$\vec{\nabla}\times\vec{V}$ vector → vector

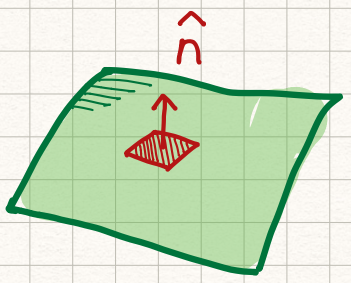
and ∇^2 operator doesn't change tensor property.

The inverse operations of these vector operators are various integrals.



line integral $\int_\Gamma \vec{F} \cdot d\vec{r}$

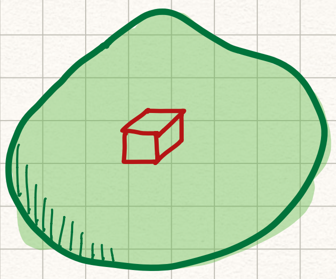
$$= \int_\Gamma F_x dx + F_y dy + F_z dz$$



$d\vec{a} = da\,\hat{n}$

surface integral $\int_S \vec{E} \cdot d\vec{a}$

$$= \int_S (\vec{E} \cdot \hat{n})\, da$$



$d\tau = dxdydz$

volume integral $\int_V \varphi\, d\tau$

$$= \int \varphi\, dxdydz$$

Helmholtz Theorem

To describe the spatial variations of a scalar field $\phi = \phi(x,y,z)$, one needs 3 derivatives

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

— gradient suits the need as expected

What about a vector field like $\vec{E}$ or $\vec{B}$?

Naively, one expects $3\times 3 = 9$ derivatives are needed here.

$$\frac{\partial E_i}{\partial x_j}, \frac{\partial B_i}{\partial x_j} \rightarrow 9+9 = 18 \text{ derivatives}$$

BUT, Maxwell does not think so, ha!

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

It seems that one only need to know ~

(1) $\vec{\nabla}\cdot\vec{F} = D(\vec{r})$

(2) $\vec{\nabla}\times\vec{F} = \vec{C}(\vec{r})$

to describe the vector field $\vec{F}(\vec{r})$.

It turns out to be true – Helmholtz theorem.

As long as $D, \vec{C} \to 0$ faster than $\frac{1}{r^2}$, the vector field $\vec{F}$ can be uniquely expressed in terms of the following decomposition:

$$\boxed{\vec{F} = -\vec{\nabla} U + \vec{\nabla} \times \vec{W}}$$

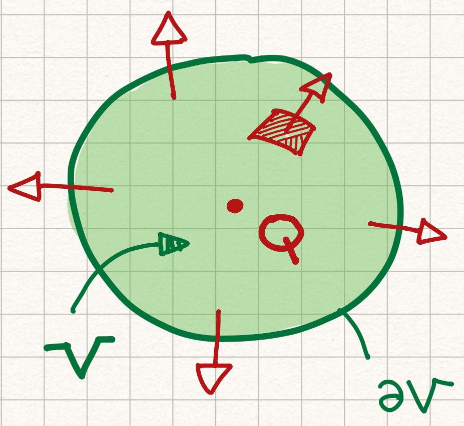
$$U = \frac{1}{4\pi} \int \frac{D(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\tau' \longleftarrow \text{Coulomb potential}$$

$$\vec{W} = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\tau'$$

Thus, as long as the divergence $D(\vec{r})$ & the curl $\vec{C}(\vec{r})$ is given, the potentials U, $\vec{W}$ can be constructed by the integrals.

→ The vector field $\vec{F}(\vec{r})$ is uniquely determined by its divergence & curl.

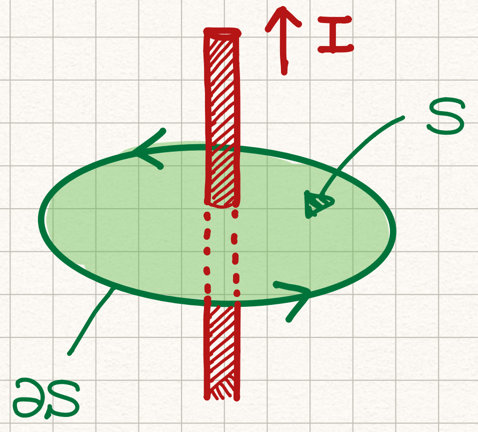
Bulk-Boundary Relations



divergence theorem

$$\int_{\partial V} \vec{E}\cdot d\vec{a} = \int_{V} \vec{\nabla}\cdot\vec{E}\, d\tau$$

boundary　　　bulk



Stokes' theorem

$$\int_{\partial S} \vec{B}\cdot d\vec{r} = \int_{S} (\vec{\nabla}\times\vec{B})\cdot d\vec{a}$$

boundary　　　bulk

boundary integral　　　bulk integral

Observations: L.H.S. = R.H.S.

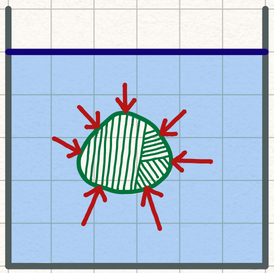
no derivative　　　"some" derivative

simple example:

$$\underbrace{F(b)-F(a)}_{\text{boundary}} = \int_a^b \underbrace{\frac{dF}{dx}\cdot dx}_{\text{bulk}}$$

Archimedes Principle

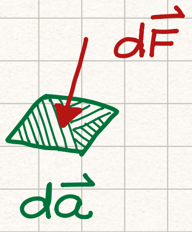
Floating force = weight of expelled liquid



In vector form,

$$\underbrace{\vec{B}}_{\text{surface integral}} = -\underbrace{\vec{W}}_{\text{volume integral.}}$$

Elaborate a bit more ...

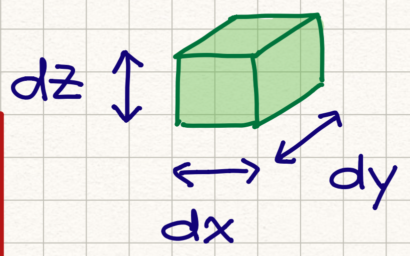


$$d\vec{F} = -\underbrace{p}_{\substack{\text{pressure in isotropic liquid} \\ \text{is a scalar ☺}}}\, da\, \hat{n} = -p\, d\vec{a}$$

$$\vec{B} = \int d\vec{F} = -\int p\, d\vec{a} \quad (\text{surface integral})$$

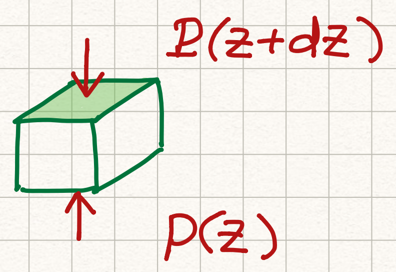
$$\vec{W} = \int d\vec{W} = \int \rho \vec{g}\, d\tau \quad (\text{volume integral})$$

Let us derive the "field equation" for the pressure $p = p(x, y, z)$:



Mr. Cube

$$\boxed{\begin{aligned} -p(z+dz)\,dxdy + p(z)\,dxdy \\ - \rho g\, dxdydz = 0 \end{aligned}}$$



$$p(z+dz) - p(z) = \frac{\partial p}{\partial z}\, dz$$

$$\longrightarrow \left(- \frac{\partial p}{\partial z} - \rho g \right) \overset{dxdydz}{d\tau} = 0$$

Thus, from $\vec{F} = m\vec{a}$, one obtains the equations

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial x} = 0 = \frac{\partial p}{\partial y} \longrightarrow \boxed{\vec{\nabla} p = \rho \vec{g}}$$

pressure looks like "potential".

Making use of the following theorem,

$$\boxed{\int p\, d\vec{a} = \int \vec{\nabla} p\, d\tau}$$

$$\vec{B} = -\int p\, d\vec{a} = -\int \vec{\nabla} p\, d\tau = -\int \rho \vec{g}\, d\tau$$

$$\longrightarrow \vec{B} = -\vec{W}$$ Achimedes principle!