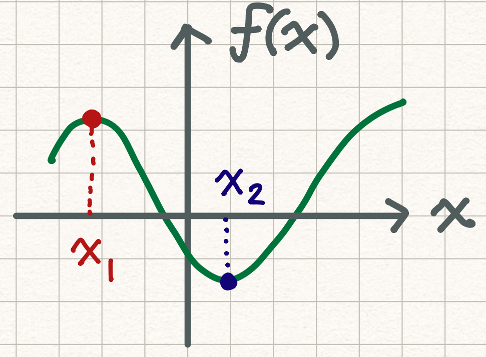
Stationary Point



x_1 : local max

x_2 : local min

One can use differential to spot the max/min of a function:

$$df = 0$$

Or, in more familiar form

$$\frac{df}{dx}dx = 0 \longrightarrow \frac{df}{dx} = 0$$

Furthemore, one can use Taylor expansion to describe the vicinity near the stationary pt.

($df = 0$

$$f(x) = f(x_0) + \cancel{f'(x_0)}^{\,0}\,\Delta x + \frac{1}{2!}f''(x_0)(\Delta x)^2 + \cdots$$

Here $\Delta x = x - x_0$. Near the stationary pt, $\Delta x \ll 1$, higher-order terms can be ignored.

$$\Delta f \approx \frac{1}{2!}f''(x_0)\,\Delta x^2$$

$f''(x_0) > 0$ min

$f''(x_0) < 0$ max

when $f''(x_0) = 0$, it can be complicated

Stationary pt of multi-variable fn.

The vicinities of stationary points of a multi-variable fn $f(x,y)$ are complicated.

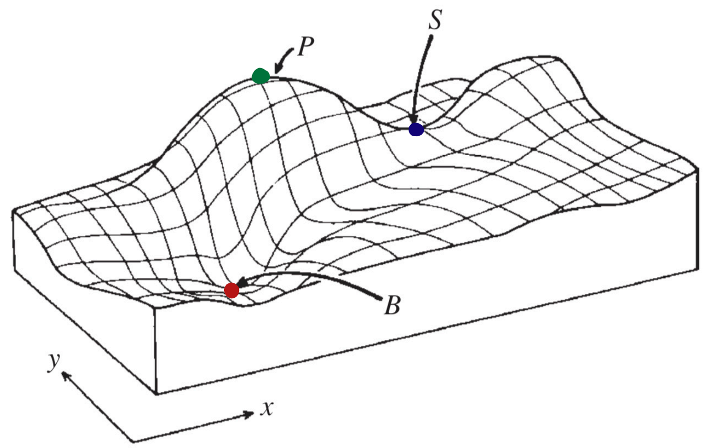


Figure 5.2 Stationary points of a function of two variables. A minimum occurs at B, a maximum at P and a saddle point at S.

Riley, Hobson & Bence (3rd edition)

min @ Basin

max @ Peak

and Saddle pt

But, the differential works equally well.

$$df = 0 \longrightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Because dx, dy are arbitrary, it means

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}}$$

Gradient can be introduced

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$df = \vec{\nabla} f \cdot d\vec{r}$$

$d\vec{r} = (dx, dy)$ arbitrary!

$$\longrightarrow \vec{\nabla} f = 0$$

There are many types of stationary pts.

How can we tell? By the 2^{nd} derivatives!

Making use of the Taylor expansion,

$$\Delta f = f(x,y) - f(x_0,y_0)$$

$$= \cancelto{0}{\sum_i \frac{\partial f}{\partial x_i}} \Delta x + \frac{1}{2!} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \cdots$$

Introduce the real and symmetric matrix M

$$M_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{OR} \quad M = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

In the vicinity of the stationary pt,

$$\boxed{\Delta f = \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j = \frac{1}{2} \Delta x^T M \Delta x}$$

stability analysis

OR, in matrix form explicitly ☺

$$\Delta f = \frac{1}{2} (\Delta x \;\; \Delta y) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

To perform the stability analysis, one needs to diagonalize the matrix M. We will learn how to do this in the linear algebra later.

Let us introduce two important properties of a matrix first.

Trace. $\operatorname{tr} M = \sum_i M_{ii} = M_{11} + M_{22}$

Determinant. $\det M = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}$

$$= M_{11} M_{22} - M_{12} M_{21}$$

B: $\det M > 0$

$\operatorname{tr} M > 0$

P: $\det M > 0$

$\operatorname{tr} M < 0$

S: $\det M < 0$

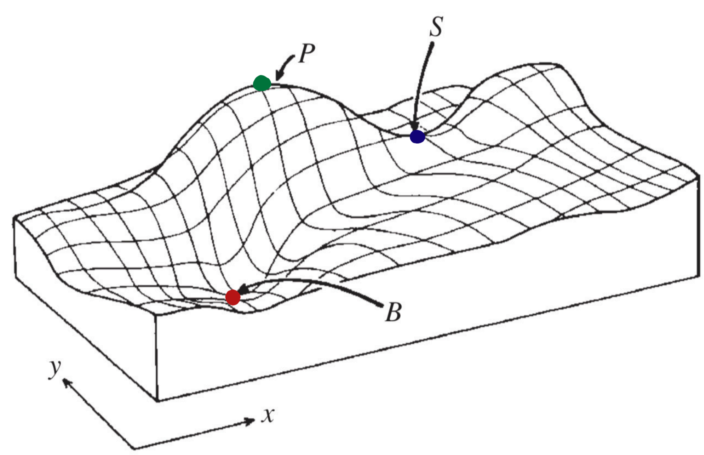


Figure 5.2 Stationary points of a function of two variables. A minimum occurs at B, a maximum at P and a saddle point at S.

Riley, Hobson & Bence (3rd edition)

Stationary Points with Constraints

Sometimes one needs to find the stationary pt of a fn $f(x,y)$ with some constraint $g(x,y)=0$

The easiest way is usually the method of **Lagrange multiplier**

Complicate the problem by adding one more variable λ !

auxiliary variable

$$f^*(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

Then, write down the stationary conditions for $f^*(x,y,\lambda)$.

$$df^* = 0 \longrightarrow \frac{\partial f^*}{\partial x}dx + \frac{\partial f^*}{\partial y}dy + \frac{\partial f^*}{\partial \lambda}d\lambda = 0$$

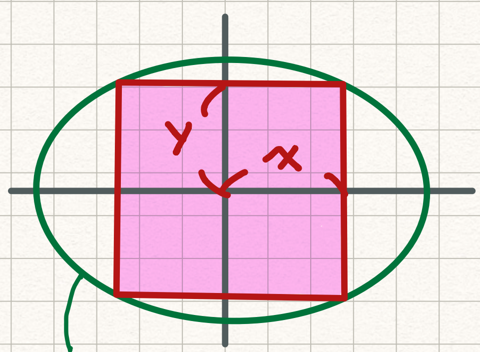
Because $dx, dy, d\lambda$ are arbitrary,

$$\frac{\partial f^*}{\partial x} = 0 \longrightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial f^*}{\partial y} = 0 \longrightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

$\frac{\partial f^*}{\partial \lambda} = 0 \longrightarrow g = 0$ — the constraint is automatically satisfied ☺ YES!

Example Find the maximum area $A(x,y) = 4xy$ of the inscribed rectangle inside the ellipse.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$A^*(x,y,\lambda) = A(x,y) + \lambda \phi(x,y)$$

where $\phi(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

So, as explained before,

$\frac{\partial A^*}{\partial \lambda} = 0$ just gives the required constraint ☺

$\frac{\partial A^*}{\partial x} = 0 \longrightarrow 4y + \frac{2\lambda}{a^2} x = 0$ — (1)

$\frac{\partial A^*}{\partial y} = 0 \longrightarrow 4x + \frac{2\lambda}{b^2} y = 0$ — (2)

Now, some algebra is in order ...

$(1) \times x + (2) \times y$ gives

$$8xy + 2\lambda \underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}_{\text{just } 1 \text{ !}} = 0$$

$$\rightarrow \quad \lambda = -4xy$$

Substitute back in (1),

$$4\cancel{y} + \frac{2}{a^2}x(-4x\cancel{y}) = 0 \rightarrow x^2 = \frac{a^2}{2}$$

$$\rightarrow \quad x = \frac{a}{\sqrt{2}}$$

Substitute λ into (2),

$$4\cancel{x} + \frac{2}{b^2}y(-4\cancel{x}y) = 0 \rightarrow y^2 = \frac{b^2}{2}$$

$$\rightarrow \quad y = \frac{b}{\sqrt{2}}$$

Thus, the maximal area of the inscribed rectangle is

$$A_{max} = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = 2ab \quad \#$$

Example revisited

Making use of parametric angle θ to eliminate the constraint:

$$x = a\cos\theta$$
$$y = b\sin\theta$$

The angular variable θ is free now !

$$A(x,y) = 4xy = 4\,a\cos\theta \cdot b\sin\theta$$
$$= 4ab\cos\theta\sin\theta$$
$$= 2ab\sin(2\theta) \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

The maximum occurs @ $\theta = \frac{\pi}{4}$

$$x = a\cos\frac{\pi}{4} = \frac{a}{\sqrt{2}}$$
$$y = b\sin\frac{\pi}{4} = \frac{b}{\sqrt{2}} \longrightarrow A_{max} = 2ab$$

Geometric interpretation of λ

Let's move into the 3D space. Without any constraint, the total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \vec{\nabla} f \cdot d\vec{r}$$

If the gradient is not zero, $\vec{\nabla} f \neq 0$, one can choose $d\vec{r}$ along with $\vec{\nabla} f$ so that df can increase or decrease.

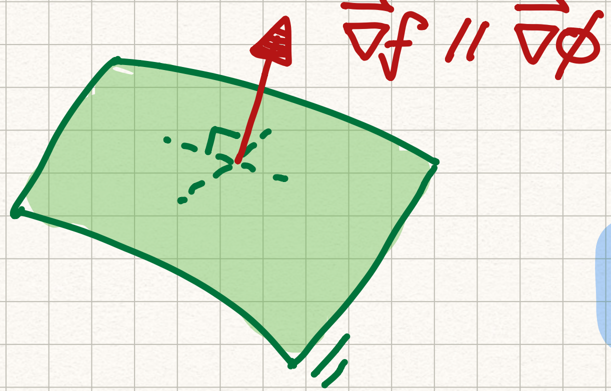
In consequence, it cannot be the local maximum nor the local minimum!

In short, the stationary pt requires $df = 0$

$$\rightarrow \quad \vec{\nabla} f \cdot \underbrace{d\vec{r}}_{\text{arbitrary}} = 0 \quad \rightarrow \quad \vec{\nabla} f = 0$$

What happens if $d\vec{r}$ is NOT arbitrary?

With the constraint $\phi(x,y,z)=0$, $d\vec{r}$ can only move on the surface!



$\boxed{df=0}$ still holds!

But, it leads to different criterion

Note that $\phi(x,y,z)=0$, so its differential is also zero: $d\phi=0$

$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = 0$ $\quad\quad$ $\boxed{d\vec{r}\cdot\vec{\nabla}\phi=0}$

The stationary condition now becomes

$\vec{\nabla}f\cdot d\vec{r}=0 \quad + \quad d\vec{r}\cdot\vec{\nabla}\phi=0$

$\rightarrow$ $\vec{\nabla}f$ cannot have any inplane component

thus, $\vec{\nabla}f \parallel \vec{\nabla}\phi$ $\rightarrow$ $\boxed{\vec{\nabla}f+\lambda\vec{\nabla}\phi=0}$

This is the geometric interpretation of the Lagrange multiplier λ!