Complex Series

The sum of the first N terms of a series is

$$S_N = z_1 + z_2 + \cdots + z_N = \sum_{n=1}^{N} z_n$$

Here z_n can be real, imaginary and complex.

example Maclaurin expansion of e^z

$$S(z) = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots\cdot$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

infinite series

Later, we will learn Taylor series and you will know that $S(z) = e^z$

In the following, we will introduce

(1) arithmetic series

(2) geometric series

(3) Taylor series — very important

arithmetic series

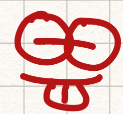
$a_n = a_0 + nd$

$$S_N = a_0 + (a_0+d) + (a_0+2d) + \cdots + [a_0 + (N-1)d]$$

$$= \sum_{n=0}^{N-1} (a_0 + nd)$$

$$= a_0 \sum_{n=0}^{N-1} 1 + d \sum_{n=0}^{N-1} n$$

$$= a_0 N + d \cdot \frac{1}{2} N(N-1)$$



a piece of cake

geometric series

$a_n = a_0 r^n$

$$S_N = a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{N-1}$$

$$= \sum_{n=0}^{N-1} a_0 r^n$$

There are many ways to carry out the sum.

$$r^N - 1 = (r-1)(\underbrace{r^{N-1} + r^{N-2} + \cdots + r + 1}_{\sum_{n=0}^{N-1} r^n})$$

$$\rightarrow \sum_{n=0}^{N-1} r^n = \frac{r^N - 1}{r - 1} = \frac{1 - r^N}{1 - r}$$

Thus, the sum of a geometric series is

$$S_N = \sum_{n=0}^{N-1} a_0 r^n = a_0 \sum_{n=0}^{N-1} r^n$$

$$= a_0 \frac{1-r^N}{1-r} = a_0 \frac{r^N-1}{r-1}$$

So, the formula seems to carry singularity @ $r=1$. Is this singularity of S_N real?

Nope, it is not real ☺

example $S = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$

Introduce the function $f(x) = 1 + 2x + 3x^2 + \cdots$

It is clear that $S = f(\frac{1}{2})$.

Now, we need to carry out the sum for $f(x)$.

$$F(x) = \int_0^x f(x')\,dx' = x + x^2 + x^3 + \cdots$$

$$= x(1 + x + x^2 + \cdots) = \frac{x}{1-x}$$

Making use of the relation $f(x) = F'(x)$,

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}\left[\frac{x}{1-x}\right]$$

$$\frac{x-1+1}{1-x} = -1 + \frac{1}{1-x}$$

$$= \frac{d}{dx}\left[-1 + \frac{1}{1-x}\right]$$

$$= \frac{1}{(1-x)^2} \longrightarrow \boxed{S = f\left(\tfrac{1}{2}\right) = 4}$$

example

$$S(\theta) = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \cdots$$

Making use of the complex algebra ☺

$$S(\theta) = \mathrm{Re}\left\{ 1 + e^{i\theta} + \frac{1}{2!}e^{i2\theta} + \frac{1}{3!}e^{i3\theta} + \cdots \right\}$$

Replacing the phase by the complex number,

$e^{i\theta} = z$, the infinite sum becomes

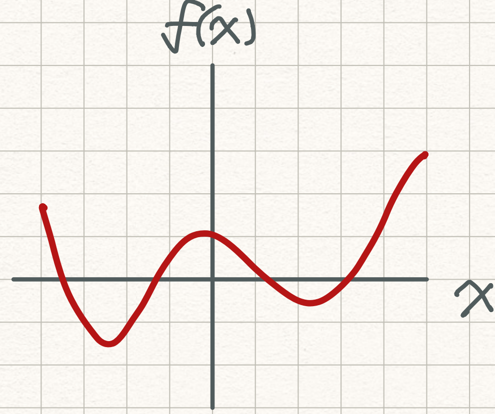
$$S(\theta) = \mathrm{Re}\left\{ 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right\}$$

$$= \mathrm{Re}\left\{ e^{z} \right\} = \mathrm{Re}\left\{ e^{\cos\theta + i\sin\theta} \right\}$$

$$= e^{\cos\theta} \cos(\sin\theta)$$

pretty cool !

Taylor Series



Given a function $f(x)$ as shown on the right. Is it possible to express it in terms of power series?

$$f(x) \stackrel{?}{=} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Skipping the math rigors, one can try to compute the coefficients a_n

(1) just plug $x=0$ into $f(x)$

$$f(0) = a_0$$ simple...

(2) take a derivative, then plug in $x=0$

$$f'(0) = a_1$$ yes!

(3) take two derivatives, then plug in $x=0$

$$f''(0) = 2a_2 \longrightarrow a_2 = \frac{f''(0)}{2!}$$

Keep taking higher-order derivatives,

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Collecting all results together, the fn $f(x)$ can be expressed as a power series.

$$\boxed{\begin{aligned} f(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}}$$

This result is remarkable ~ the local properties of the fn $[f(0), f'(0), f''(0) \cdots$ all around $x=0]$ dictates its global profile !

example $\qquad f(x) = e^x$

$f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots, \quad f^{(n)}(x) = e^x, \quad \cdots$

$\longrightarrow \quad f(0) = f'(0) = f''(0) = \cdots = 1$

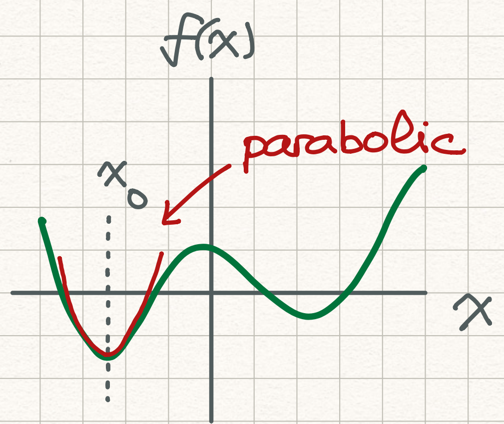
thus, $\quad e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

One can apply the same trick to expand $f(x)$ at an arbitrary point $x = x_0$

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

It is straightforward to show that $a_n = \dfrac{f^{(n)}(x_0)}{n!}$

$$\boxed{\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \end{aligned}}$$



One can expand $f(x)$ @ its equilibrium pt [$f'(x_0) = 0$]

According to Taylor series

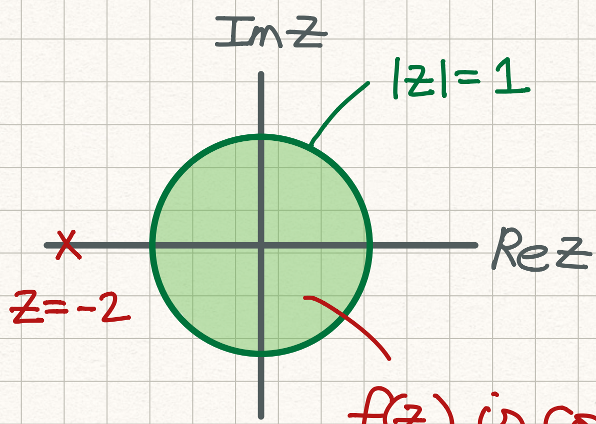
$$f(x) = f(x_0) + \cancelto{0}{f'(x_0)}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$\approx f(x_0) + \underbrace{\frac{f''(x_0)}{2!}(x-x_0)^2}_{\text{quadratic potential}}$$

Analytic Continuation

All the series properties can be generalized to complex numbers on the complex plane.

For instance, consider the infinite sum $f(z)$



$$f(z) = 1 + z + z^2 + \cdots$$

It can be shown that $f(z)$ is convergent for $|z| < 1$.

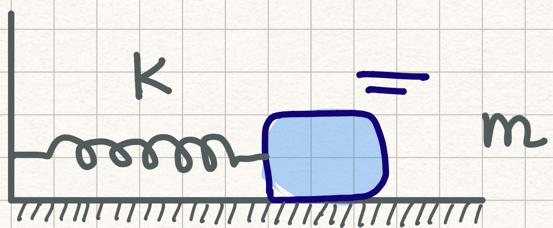
So, does it make any sense when $z = -2$?

$$f(-2) = 1 - 2 + 4 - 8 + \cdots$$

If I tell you that $1 - 2 + 4 - 8 + \cdots = \frac{1}{3}$,

maybe you will just drop the applied math course ...

simple harmonic oscillator



The mass is worn out a bit $m = m_0 - \Delta m$

How is the period modified?

Assuming the mass difference is small

$$g = \frac{\Delta m}{m_0} = 1 - \frac{m}{m_0} \ll 1$$

The equation of motion is

$$(m_0 - \Delta m)\ddot{x} + kx = 0$$

Making everything dimensionless

$$\omega_0^2 = \frac{k}{m_0} \quad \text{so set} \quad \omega_0 t \to t$$

and introduce $\alpha = \frac{\omega^2}{\omega_0^2}$

The EOM becomes dimensionless

$$\ddot{x} + x - g\ddot{x} = 0$$

perturbation!

The solution $x(t)$ can be expanded,

$$x(t) = x_0(t) + g x_1(t) + g^2 x_2(t) + \cdots$$

The zero-th order solution $x_0(t)$ takes the form

$$x_0(t) = \cos(\sqrt{\alpha}\, t) \qquad \alpha = 1 + c_1 g + c_2 g^2 + \cdots$$

Let's compute the coefficient c_n !

①
$$\begin{aligned}(1-g)\ddot{x} &= -\alpha(1-g)\cos(\sqrt{\alpha}\, t) \\ &\quad + (1-g)(g\ddot{x}_1 + g^2\ddot{x}_2 + \cdots) \\ &= -\cos(\sqrt{\alpha}\, t) \\ &\quad + g\left\{(1-c_1)\cos(\sqrt{\alpha}\, t) + \ddot{x}_1\right\} \\ &\quad + g^2\left\{(c_1 - c_2)\cos(\sqrt{\alpha}\, t) + (\ddot{x}_2 - \ddot{x}_1)\right\} \\ &\quad + \circ\circ\circ\end{aligned}$$

② $$x(t) = \cos(\sqrt{\alpha}\, t) + g x_1 + g^2 x_2 + \cdots$$

Now, write down EOM order by order

zero-th order :

trivial.

$$-\cos(\sqrt{\alpha}t) + \cos(\sqrt{\alpha}t) = 0$$

1st order :

$$\ddot{x}_1 + x_1 + (1-q)\cos(\sqrt{\alpha}t) = 0$$

$$\rightarrow \quad x_1(t) = A\cos(t+\phi) + B\cos(\sqrt{\alpha}t)$$

unknown constants

substitute back into EOM.

$$(1-\alpha)B\cos(\sqrt{\alpha}t) + (1-q)\cos(\sqrt{\alpha}t) = 0$$

$$\underbrace{(1-\alpha)}_{O(g)}B + \underbrace{(1-q)}_{O(1)} = 0$$

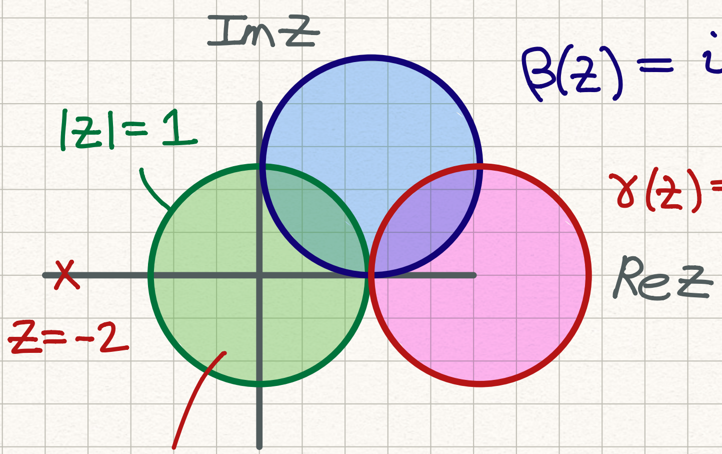
Thus, $q = 1$ and $B = 0$

Because $x_1(0) = 0$ and $\dot{x}_1(0) = 0$,

$A = 0$ and $\phi = 0 \rightarrow x_1(t) = 0$

One can work out the calculations to all orders.

$$\alpha = \frac{\omega^2}{\omega_0^2} = 1+g+g^2+\cdots = \sum_{n=0}^{\infty} g^n \quad \frac{1}{1-g}$$



$\alpha(z)$ is convergent!

$$\beta(z) = i\left\{1+i(g-1-i)+i^2(g-1-i)^2+\cdots\right\}$$

$$\gamma(z) = -1\left\{1-(g-2)+(g-2)^2+\cdots\right\}$$

$$\Downarrow$$

$$f(z) = \frac{1}{1-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^n$$

By analytic continuation on the complex plane,

$$\alpha(z)=\beta(z)=\gamma(z) = \frac{1}{1-z}$$

thus, $\alpha(z=-2) = 1-2+4-8+\cdots$

$$= \frac{1}{1-(-2)} = \frac{1}{3}$$

Does this make sense? $m = m_0 - \Delta m = (1-g)m_0$

For $g=-2$, $m=(1+2)m_0 = 3m_0$

$$\frac{\omega^2}{\omega_0^2} = \frac{K/m}{K/m_0} = \frac{m_0}{m} = \frac{1}{3}$$

YES