Partial Differentiation

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The notes cover the introduction to partial differentiation (Boas 4.1-4.4).

• Definition of partial differentiation

Consider a function $z = f(x, y)$ and it corresponds to a surface in the three dimensional space. At a particular point (x_0, y_0), can one find the "slope" as for the one-variable function? This question is indeed quite interesting and motivates us to introduce the *partial derivatives*,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{1}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{2}$$

It is important to emphasize that, when taking partial derivative with respect to x, the other variable y is held constant. This is the central key to the notion of partial differentiation. For example, consider the following function and its equivalent expressions in Cartesian and polar coordinates,

$$z = x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = 2x^2 - r^2 = r^2 - 2y^2.$$

By holding different variables constant, the partial derivatives with respect to r are also different,

$$\left(\frac{\partial z}{\partial r}\right)_\theta = 2r(\cos^2\theta - \sin^2\theta),$$

$$\left(\frac{\partial z}{\partial r}\right)_x = -2r,$$

$$\left(\frac{\partial z}{\partial r}\right)_\theta = 2r.$$

Thus, the partial differentiation only makes sense when one specifies which variable is held constant.

• Taylor expansion with multiple variables

Following the same spirit, one can construct the Taylor expansion for the function $f(x, y)$ around the point (a, b). Write the function as a power series of $(x - a)$ and $(y - b)$,

$$\begin{aligned} f(x,y) &= a_{00} + \Big[a_{10}(x-a) + a_{01}(y-b)\Big] \\ &\quad + \Big[a_{20}(x-a)^2 + a_{11}(x-a)(y-b) + a_{02}(y-b)^2\Big] + ... \end{aligned}$$

Introduce the short-hand notation $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. It is straightforward to show that $a_0 0 = f(a, b)$, $a_{10} = f_x(a, b)$ and $a_{01} = f_y(a, b)$. The coefficients of the higher-order terms can be found in the similar way. Collecting all pieces together, the Taylor expansion can be cast into the suggesting form,

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^n f(a,b) = e^{h\partial/\partial x + k\partial/\partial y} f(a,b), \tag{3}$$

where $h = x-a$ and $k = y-b$. It is quite interesting that the Taylor expansion for $f(x, y)$ can be view as acting an exponential operator on $f(a, b)$.

• Total differentials

An important concept in multiple-variable differentiation is *total differential*. Consider the change of $f(x, y)$ when both variable change,

$$\begin{aligned} \Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= [f(x+\Delta x, y) - f(x,y)] + [f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)]. \end{aligned}$$

Taking both Δx and Δy to be infinitesimal, the above equation becomes

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \tag{4}$$

The above equation contains all information you need to know. For instance, if y is held constant ($dy = 0$), the equation reduces to

$$\left(\frac{df}{dx} \right)_y = \frac{\partial f}{\partial x}.$$

This is exactly the definition for partial derivative mentioned in previous paragraphs.

• Error estimate

Differential is a very useful tool to estimate errors. For instance, the resistance for a cylindrical wire is proportional to its length but inverse proportional to the square of the radius,

$$R = \rho \frac{L}{r^2}, \tag{5}$$

where ρ is some constant depending on the material properties. Suppose the error in measuring length and radius is about 5%, what is the maximal resultant error in resistance? This question can be answered easily by taking the total differential,

$$dR = \rho \frac{1}{r^2} dL - 2\rho L \frac{1}{r^3} dr.$$

Divide the above relation by the resistance R, we obtain the celebrated relation between different kinds of errors,

$$\frac{dR}{R} = \frac{dL}{L} - 2\frac{dr}{r}.$$

In the worse case, the errors in length and radius take opposite sign and add up. Thus, the maximal error in resistance is

$$\left| \frac{dR}{R} \right|_{max} = \left| \frac{dL}{L} \right| + 2 \left| \frac{dr}{r} \right|,$$

which can be as large as 15%! The error in resistance is enhanced by three times from the original error in length and radius.