Vectors, Lines and Planes

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The notes cover vectors, lines and planes (Boas 3.4-3.5). I will review the basic properties of vectors including inner and outer products between two vectors. In Cartesian coordinates, the vector products can be expressed in terms of decomposed components. Lines and planes can be expressed by appropriate outer and inner products. Finally, the important notation of linear independence is introduced and its connection to determinant is explained.

• Vectors

A vector contains two important pieces of information – length (magnitude) and direction. I shall not bore you with the addition/substraction of vectors, which should be taught in high-school math. Unlike ordinary numbers, there are two kinds of products between two vectors. The scalar product (inner product) is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta, \tag{1}$$

where θ is angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ and $|\boldsymbol{A}|, |\boldsymbol{B}|$ denote the lengths of the corresponding vectors. The vector product, also known as outer product, is slightly more complicated,

$$\boldsymbol{A} \times \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \sin\theta\, \hat{\boldsymbol{e}}_{AB}, \tag{2}$$

where $0 \geq \theta < \pi$ is the positive angle between $\boldsymbol{A}$ and $\boldsymbol{B}$. The unit vector $\hat{\boldsymbol{e}}_{AB}$ follows the right-hand rule and is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$.

• Vectors in Cartesian coordinates

In Cartesian coordinates, a vector can be represented by its components

$$\boldsymbol{V} = V_x\boldsymbol{i} + V_y\boldsymbol{j} + V_z\boldsymbol{k}, \tag{3}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the (positive) direction of x-, y- and z-axes. The inner products of these base vectors follow the relations,

$$\begin{aligned} \boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1, \\ \boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0. \end{aligned}$$

By distributive law, the inner product can be expressed in terms of Cartesian components,

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z. \tag{4}$$

Similarly, we can work out the relations for outer products between the base vectors,

$$\begin{aligned}
\boldsymbol{i}\times\boldsymbol{j} &= -\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k},\\
\boldsymbol{j}\times\boldsymbol{k} &= -\boldsymbol{k}\times\boldsymbol{j} = \boldsymbol{i},\\
\boldsymbol{k}\times\boldsymbol{i} &= -\boldsymbol{i}\times\boldsymbol{k} = \boldsymbol{j},\\
\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} &= \boldsymbol{k}\times\boldsymbol{k} = 0.
\end{aligned}$$

These relations can be elegantly expressed in terms of the Levi-Civita symbol,

$$\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = \epsilon_{ijk}\hat{\boldsymbol{e}}_k, \tag{5}$$

where $\hat{\boldsymbol{e}}_1 = \boldsymbol{i}$, $\hat{\boldsymbol{e}}_2 = \boldsymbol{j}$ and $\hat{\boldsymbol{e}}_3 = \boldsymbol{k}$. The distributive laws thus gives the important formula,

$$(\boldsymbol{A}\times\boldsymbol{B})_k = \sum_{i,j=1}^{3} \epsilon_{ijk} A_i B_j. \tag{6}$$

Or, it can be written as a determinant

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_k \\ B_x & B_y & B_z \end{vmatrix}. \tag{7}$$

• Lines

We can make use to the inner and outer products to describe lines and planes. Suppose $\boldsymbol{r}_0$ is a point on a particular straight line. An arbitrary point $\boldsymbol{r}$ on the line must satisfy the geometric constraint,

$$(\boldsymbol{r}-\boldsymbol{r}_0)\times\hat{\boldsymbol{e}} = 0, \tag{8}$$

where $\hat{\boldsymbol{e}}$ is the unit vector along the direction of the straight line. It is important that the constraint is purely geometrical and is independent of coordinate choices. In Cartesian coordinates, the above constraint can be brought into the form,

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}, \tag{9}$$

where (a, b, c) is a vector (not necessarily a unit vector) parallel to $\hat{\boldsymbol{e}}$.

• Planes

Suppose $\boldsymbol{r}_0$ is a point on a particular plane. An arbitrary point $\boldsymbol{r}$ on the plane must satisfy the geometric constraint,

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \hat{\boldsymbol{n}} = 0, \tag{10}$$

where $\hat{\boldsymbol{n}}$ is the normal vector of the plane. Again, the constraint is geometric and stands without any introduction of specific coordinates. In Cartesian coordinates, the above constraint can be casted into

$$ax + by + cz = d, \tag{11}$$

where (a, b, c) is a vector (not necessarily a unit vector) parallel to $\hat{\boldsymbol{n}}$.

• Linear independence and rank

For a set of n vectors, they are called *linearly dependent* if the following constraint

$$k_1 \boldsymbol{V}_1 + k_2 \boldsymbol{V}_2 + ... + k_n \boldsymbol{V}_n = 0 \tag{12}$$

allows a non-trivial solution for k_i with $k_1^2 + k_2^2 + ... + k_n^2 \neq 0$. One the other hand, if the above constraint is only satisfied with the trivial solution $(k_1, k_2, ..., k_n) = (0, 0, ..., 0)$, these vectors are called *linearly independent*.

Take $n = 3$ as an example. We can introduce the coefficient matrix

$$M = \begin{pmatrix} V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \\ V_{3x} & V_{3y} & V_{3z} \end{pmatrix} \tag{13}$$

to rewrite the constraint in a more familiar form

$$M\boldsymbol{k} = 0, \quad \rightarrow \quad \begin{pmatrix} V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \\ V_{3x} & V_{3y} & V_{3z} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{14}$$

Obviously, $(k_1, k_2, k_3) = (0, 0, 0)$ is a solution. Therefore, if $\det M \neq 0$, this would be the only solution and the vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are linearly independent. If $\det M = 0$, there are infinite solutions and it is possible to find non-trivial solutions, satisfying the requirement $k_1^2 + k_2^2 + ... + k_n^2 \neq 0$. In this case, the three vectors are linearly dependent. It is very cute that we can use the determinant constructed from the three vectors to figure out whether they are linearly independent or not.