Matrix Operations

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The notes cover matrix operations (Boas 3.6). I will go through the basic matrix operations and also touch upon the notion of commutators, functions of matrices and so on.

• Basic Operations

Additions and subtractions of matrices are simple. Multiplication between matrices is more complicated and can be written in the algebraic form,

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}. \tag{1}$$

From the definition, it is clear that the column number of A and the row number of B must equal. Otherwise, the summation over the index k is not well defined. That is to say, not all matrices can be multiplied together!

It's often convenient to define the *transpose* of a matrix,

$$(A^T)_{ij} = A_{ji}, \tag{2}$$

by switching the rows and columns. For a $m \times n$ matrix, its transpose is an $n \times m$ matrix. For a square matrix, it remains a square matrix of the same size. If $A^T = A$, the matrix is called symmetric. It is interesting to mention that the determinant remains invariant under the operation of transpose,

$$\det(A^T) = \det A. \tag{3}$$

In addition, if both A and B are square matrices of the same dimension,

$$\det(AB) = \det A \det B. \tag{4}$$

The above identity is quite useful. For instance, choose $A = k\mathbf{1}$,

$$\det(kB) = \det(k\mathbf{1}) \det B = k^n \det B.$$

• Commutators

Suppose we choose two matrices to compute their product,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The answer depends on the order of multiplication,

$$\begin{aligned} AB &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ BA &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Observe that AB is not the same as BA an we say that A and B do not *commute*. The commutator between A and B is

$$[A, B] = AB - BA. \tag{5}$$

In general, the commutator between two square matrices (of the same dimension) is not zero, marking the unique feature of matrix multiplication.

The non-commuting properties are extremely important concept in quantum theory. For instance, the momentum of a particle can be viewed as the differential operator,

$$p \to -i\hbar \frac{d}{dx}.$$

Starting from the relation $(d/dx)[xf(x)] = x(d/dx)f + f(x)$, one can obtain the operator identity,

$$[x, d/dx] = -1, \quad \to \quad [x, p] = i\hbar. \tag{6}$$

The non-trivial commutator between x and p explains the uncertainty principle between these two observables.

The commutator exhibits interesting algebra as well. It is straightforward to show that the commutator for matrix products can be decomposed by the following rules,

$$[A, BC] = [A, B]C + B[A, C], \quad [AB, C] = A[B, C] + [A, C]B. \tag{7}$$

Similarly, one can prove the Jacobi identity among three matrices,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{8}$$

• Inverse Matrix

To solve the linearly coupled equations, we need to invert the matrix. The inverse of a matrix is defined as

$$MM^{-1} = M^{-1}M = \mathbf{1}. \tag{9}$$

If a matrix has an inverse, it is invertible; if it doesn't have an inverse, it is called singular. It is quite nice that the inversion of a matrix can be done by finding the cofactor,

$$M^{-1} = \frac{1}{\det M} C^T. \tag{10}$$

Here is an explicit example to get you familiar with the formula,

$$M = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 0 \\ 1 & -3 & 2 \end{pmatrix} \quad \rightarrow \quad C_{12} = (-1)^{1+2} \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = 4.$$

One can carefully work out the other matrix elements in the cofactor

$$C = \begin{pmatrix} 6 & 4 & 3 \\ 3 & 3 & 3 \\ 3 & 2 & 3 \end{pmatrix} \text{ and the inverse is } M^{-1} = \frac{1}{3} \begin{pmatrix} 6 & 3 & 3 \\ 4 & 3 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

You may be amazed why the formula works. Here is some hint to the formal proof. Recall that the determinant of a matrix can be written down by Laplace decomposition. Choose the first row for the expansion,

$$\det M = \sum_i M_{1i} C_{1i} = \sum_i M_{1i} (C^T)_{i1}.$$

The above relation establishes the equality of the first row and the first column of the matrix equation $MM^{-1} = \mathbf{1}$.

• Rotation Matrix

Rotations in two dimensions can be written in matrix form,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{11}$$

The determinant of the rotation matrix is $\cos^2\theta + \sin^2\theta = 1$. *Do you know why the determinant of a rotation matrix is always one disregarding the angle of rotation?* Try to multiply two rotation matrices together,

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}.$$

It is beautiful that the rule for matrix multiplication provides the correct intuition we know about rotations,

$$R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2). \tag{12}$$

• Functions of Matrices

The angular momentum along the z-axis can be expressed as a matrix,

$$L_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and its square is } \quad L_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}.$$

Consider a function of L_z which can be expanded in Taylor series. We can simplify the function by making use of the above relation,

$$f(L_z) = 4 + 3L_z + 2L_z^2 + L_z^3 = 6 + 4L_z = \begin{pmatrix} 6 & -4i \\ 4i & 6 \end{pmatrix}.$$

It is quite interesting that the matrix for angular momentum L_z is directly related to the rotation matrix $R(\theta)$,

$$\begin{aligned} e^{-iL_z\theta} &= 1 + (-iL_z\theta) + \frac{(-iL_z\theta)^2}{2!} + \frac{(-iL_z\theta)^3}{3!} + \frac{(-iL_z\theta)^4}{4!} + \frac{(iL_z\theta)^5}{5!} + ... \\ &= \mathbf{1}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right) - iL_z\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}...\right) \\ &= \mathbf{1}(\cos\theta) - iL_z(\sin\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

The above matrix identity points out the fundamental relation between the rotation and its corresponding angular momentum.