Complex Functions

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The notes introduce the elementary complex fucntions (Chap 2, Sections 11-16 in Boas). In particular, we would encounter the multi-valued logarithmic function and learn to live with its indefinite values. Then, we can sensibly answer what a complex root like $(1+i)^{1-i}$ means. At the end, we apply complex algebra to find the interference pattern for the multiple-slip experiment.

• Elementary functions

Starting from our good old friend, the exponential function can be decomposed into two parts,

$$e^z = e^x(\cos y + i \sin y). \tag{1}$$

It is important to emphasize that the complex exponential function is not monotonic anymore. If you walk along the imaginary axis, the exponential function is basically the sinusoidal function. Similarly, one can generalize the trigonometric functions in the entire complex plane,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tag{2}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{3}$$

It should be straightforward to convince yourself that both $\sin z$ and $\cos z$ are not bounded anymore. However, the identity $\sin^2 z + \cos^2 z = 1$ remains valid, going beyond its original geometric meaning.

• Multi-valued logarithmic function

As we proved in the previous lecture,

$$e^{z_1} e^{z_2} = e^{z_1+z_2},$$

it implies that its inverse function satisfies the relation

$$\ln z_1 + \ln z_2 = \ln(z_1 z_2).$$

In the polar representation, the logarithmic function on the complex plane is

$$\ln z = \ln r + i(\theta + 2n\pi), \tag{4}$$

where $n = 0, \pm 1, \pm 2, ...$ is an arbitrary integer. This means that the logarithmic function is not single-valued. For instance, taking logarithm of the simple integer $z = -1$, you end up with infinite imaginary numbers,

$$\ln(-1) = \ln 1 + i(2n+1)\pi = \pm\pi i, \pm 3\pi i, \pm 5\pi i, ...$$

• Complex powers and roots

With the help of complex logarithmic function, we can compute general complex powers/roots defined as

$$a^b = e^{b \ln a}. \tag{5}$$

Since $\ln a$ is multiple valued, the power a^b is multiple valued as well unless the principle value is specified. As an example, one can compute the complex power of a complex number,

$$(1+i)^{1-i} = \sqrt{2} e^{\pi/4} \left[\cos(\pi/4 - \ln\sqrt{2}) + i \sin(\pi/4 - \ln\sqrt{2}) \right] e^{2n\pi},$$

where n is an arbitrary integer.

• Multiple-slit interferences

The complex algebra helps tremendously in some physical examples. Consider the N-slip interference experiment. One needs to sum up all amplitudes first and squares the total amplitude to get the intensity pattern. However, summing a bunch of sinusoidal functions is not trivial. In this case, it is better to introduce the complex amplitude. The summation is easy in complex analysis,

$$\begin{aligned} \mathcal{A}_{tot} &= A_0 e^{i\omega t} + A_0 e^{i(\omega t + \delta)} + ... + A_0 e^{i[\omega t + (N-1)\delta]} \\ &= A_0 e^{i[\omega t + (N-1)\delta/2]} \frac{\sin(N\delta/2)}{\sin(\delta/2)}, \end{aligned} \tag{6}$$

where A_0 is the amplitude for each slip, N is the number of slips and δ is the phase shift between adjacent slips. Taking the imaginary part of the above expression, the physical amplitude is

$$A_{tot} = A_0 \sin\left(\omega t + \frac{N-1}{2}\delta\right) \frac{\sin(N\delta/2)}{\sin(\delta/2)}. \tag{7}$$

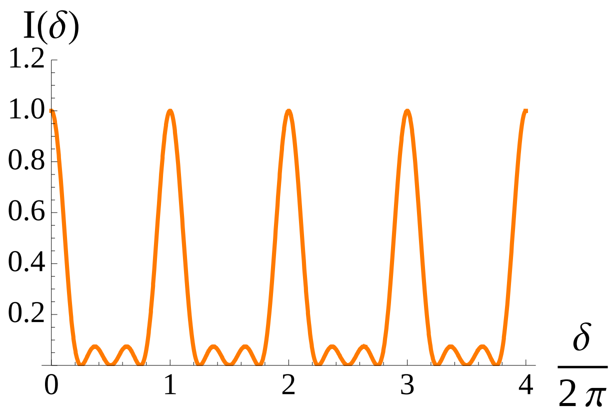

Figure 1: The intensity plot for the 4-slip interference pattern.

Square the amplitude and average over time and we obtain the interference pattern for the total intensity,

$$I(\delta) = \frac{A_0^2}{2} \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}. \tag{8}$$

The result for $N = 4$ is shown in Fig. 1. As expected, the intensity is periodic when the phase shift δ winds 2π. Within each period, there are one major peak and $N - 2$ minor peaks. A special case occurs for double-slip interferences – no minor peaks as learned in high-school physics.