Complex Series

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The notes focus on complex series (Chap 2, Sections 6-10 in Boas) and its relation to elementary functions of complex variables.

• Infinite series

Consider the following infinite series

$$S(z) = 1 - z + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} + ... \tag{1}$$

How can we know whether the infinite series S is convergent? The simplest method is to compare with the well-known geometric series, i.e. the ratio test. For the series S, the ratio test gives

$$\rho = \lim_{n\to\infty} \left| \frac{zn}{n+1} \right| = z.$$

Thus, the series is convergent for $|z| < 1$. This method can be generalized to more complicate series and helps us find out the radius of convergent.

• Complex functions in series

Elementary functions of complex variable can often be expanded by complex series. For instance, the Taylor expansion for exponential function is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... = e^x(\cos y + i \sin y). \tag{2}$$

Make us of ratio test, one can show that the series is convergent for all z. As a side remark, the Taylor expansion does not always converges to the original function. For instance, the tunneling probability through a finite barrier is

$$P \sim e^{-2\alpha\Delta x/\hbar}, \tag{3}$$

where $\alpha = \sqrt{2m(V_0 - E)}$. In the classical limit ($\hbar \to 0$), the probability goes to zero as expected. However, the tunneling effect can never be reached by classical expansions. Writing the tunneling probability as

$$P(t) = e^{-1/t}$$

with $t = \hbar/(2\alpha\Delta x)$. It is straightforward to show that each single term in the Taylor expansion for $P(t)$ is zero but the original function is certainly not zero!

• DeMoivre's theorem

In the previous lecture, we introduce Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. Using the rule of multiplication, the DeMoivre's theorem can be derived,

$$e^{in\theta} = \cos(n\theta) + \sin(n\theta) = (\cos\theta + i\sin\theta)^n. \tag{4}$$

This theorem is quite useful for deriving trigonometric identities. For instance, choose $n = 3$ in the DeMoivre's theorem and compare the real parts on both sides,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

The theorem also provides a natural way to compute roots of complex numbers,

$$z^{1/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + \sin\frac{\theta}{n}\right). \tag{5}$$

However, subtleties arise for complex roots – there are more than one possible values. Therefore, when computing the n-th root of a complex numbers, there are n possible values.