Complex Numbers – the basics

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The notes cover the basic definitions and properties of complex numbers (Boas 2.1-2.5). The story starts from finding solutions for the simple algebraic equation $x^2 + 1 = 0$. There is no real solution to the equation. But, if we introduce the notion of imaginary numbers,

$$i \equiv \sqrt{-1}, \qquad \text{with} \quad i^2 = -1, \tag{1}$$

one can write down the solutions $x = \pm i$. Going beyond the pure imaginary numbers, one can introduce the complex number with real and imaginary parts,

$$z = x + iy, \tag{2}$$

where x and y are real numbers, representing the real and imaginary parts respectively.

• Complex land looks different

For complex numbers, an amazing identity arises

$$e^{i\pi} = -1.$$

It is quite remarkable that two irrational numbers e and π can be related by the magic number i! One may question such beauty is only constrained within the territory of mathematics. This is not so. Consider light reflected by the mirror in one dimension. The incoming and outgoing waves can be written as

$$\begin{aligned} \Psi_{in}(x,t) &= A_{in}\cos(kx - \omega t), \\ \Psi_{out}(x,t) &= A_{out}\cos(kx + \omega t), \end{aligned}$$

where k is the wave number (real!) and ω is the angular frequency (also real!). But, what happens inside the wall? In fact, the wave becomes evanescent and decays exponentially,

$$\Psi_{inside}(x) \sim e^{-\alpha x}.$$

Skipping the detail derivations, the main physics is – the wave number becomes imaginary $k = i\alpha$!

• Complex plane

It is convenient to plot $z = x + iy$ in the two-dimensional complex plane, using the real part x and the imaginary part y as Cartesian coordinates. Clearly, one can also write the same complex number in *polar form*,

$$z = x + iy = r\cos\theta + ir\sin\theta. \tag{3}$$

Here $r = \sqrt{x^2 + y^2} = |z|$ is the absolute value of z (distance to the origin) and $\theta = \tan^{-1}(y/x)$ is the corresponding angle. Complex conjugate is defined as mirror mapping to the $x-$axis, i.e.

$$\bar{z} = x - iy = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)]. \tag{4}$$

Since we can plot a complex number in the two dimensional plane, a natural question pops out: Is a complex number a two-dimensional vector? Compute the following product

$$\bar{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1).$$

You can convince yourself that the real part is just the inner product and the imaginary part is the outer product for two-dimensional vectors.

• Taylor expansions

Now I would like to establish an important identity

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{5}$$

The easiest way to prove the above identity is through Taylor expansions,

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \end{aligned}$$

Choose the variable $x = i\theta$ in the Taylor expansion for the exponential function and one can check that it equals the sum of the Taylor series for the sinusoidal functions.

• Complex equations

Since a complex number contain real and imaginary parts, a complex equation amount to two real equations. For instance, $z^2 = 2i$ can be decomposed into two equations,

$$\begin{aligned} x^2 - y^2 &= 0 \\ 2xy &= 2. \end{aligned}$$

Note that both x and y are real. Thus, two solutions $x = y = 1$ and $x = y = -1$ are found for the complex equation $z^2 = 2i$.