Chapter 1 First-Order Differential Equations

1.1 Preliminary Concepts

◆ Differential equation: An equation containing the derivatives of one or more dependent variables, e.g., y', $\frac{dy}{dx}$, y_x, $\dot{y}$, with respect to one or more independent variables, e.g., x, is said to be a differential equation (D.E.)

■ Ordinary D.E.: Single independent variable

$$y' + 5y = 3x$$

■ Partial D.E.: Multiple independent variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

◆ Order (階): The order of a differential equation is the order of the highest derivative that appears in the equation

$ay'' + by' + cy = f(x)$ (second-order)

◆ Degree (次): The power of the highest derivative

$(y')^2 + y = e^x$ (degree of 2）

◆ Linearity: D.E. is linear if it has the form of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x)$$

■ Linear: 1. Degree of 1 for y and its derivatives

2. Equation contains no $y \times (y')$ term

3. Equation contains no nonlinear function of y

■ Nonlinear: Others

$(y')^2 + y = e^x$； $(1-y)y' + 2y = \sin x$； $y'' + \sin y = 0$

◆ 解（Solution）之定義

凡是能夠滿足某常微分方程式 $F(x, y, y') = 0$ 之函數 $y = y(x)$，稱為此常微分方程式之解

◆ 解之分類：

（1）通解（General solution）：通解即為常微分方程式之原始函數

（2）特解（Particular solution）：給定通解中之任意常數之值所得之解

（3）異解（Singular solution）：無法由通解中給定任意常數之值，而仍能滿足原常微分方程式之解

◆ 通解之表示方式:

（1）顯(explicit)函數表示式：

若通解之型式為 $y = f(x, c_1, c_2, ..., c_n)$

（2）隱(implicit)函數表示式：

若通解之型式為 $\phi(x, y, c_1, ..., c_n) = 0$

◆ 由通解中給定任意常數之值以決定特解時，其給定方式通常為初始條件

初始條件：針對同一個 x 值給定 y 及其導數之相應值，稱之為初始條件（Initial condition）

Ex1: Consider the initial value problem

$$y' + y = 2;\ y(1) = -5$$

［解］：The general solution of $y' + y = 2$ is $y = 2 + ke^{-x}$

$$y(1) = 2 + ke^{-1} = -5$$
$$k = -7e$$
$$y = 2 - 7ee^{-x} = 2 - 7e^{-(x-1)}$$

As a check, $y(1) = 2 - 7 = -5$

◆ 解之幾何意義：

（1）在幾何上，通解表示$x-y$平面上之某曲線族；而特解表示通解之曲線族中的某一條曲線

（2）在幾何上，異解表示通解之曲線族的包絡線

Ex2：微分方程式$(\frac{dy}{dx})^2 - x(\frac{dy}{dx}) + y = 0$，而$y = cx - c^2$，$y = x - 1$，及$y = \frac{1}{4}x^2$皆為其解

[解]：1.可經由代入驗證$y = cx - c^2$，$y = x - 1$及$y = \frac{1}{4}x^2$皆為

$$(\frac{dy}{dx})^2 - x(\frac{dy}{dx}) + y = 0 \text{ 之解}$$

2.根據定義知：

$y = cx - c^2$　　為原 O.D.E.之通解

$y = x - 1 \quad (c = 1)$　　為原 O.D.E.之特解

$y = \frac{1}{4}x^2$　　為原 O.D.E.之異解

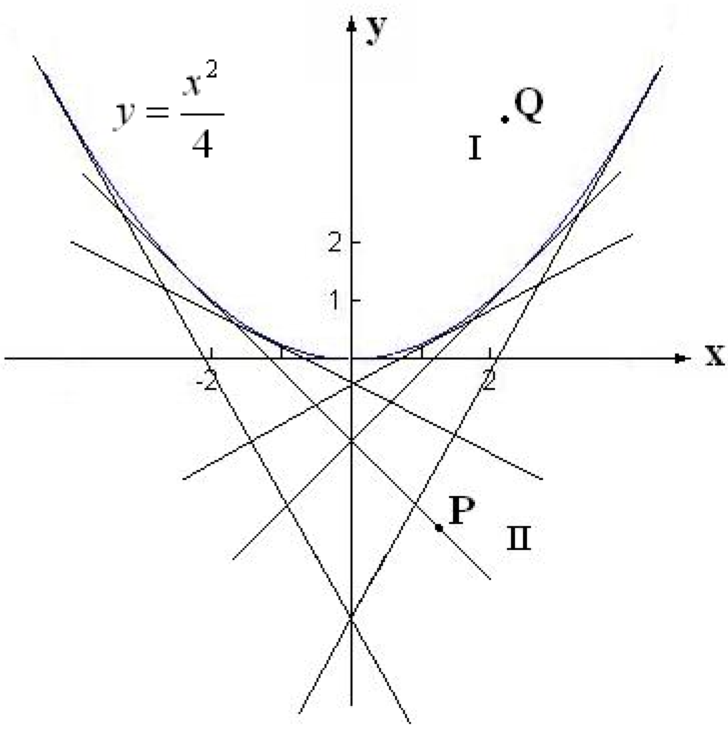
3.由下圖知

$y=cx-c^2$ 表示 $y=\frac{1}{4}x^2$ 所有切線所成之集合，

$y=x-1$ 為此集合中之一直線，而 $y=\frac{1}{4}x^2$ 為

$y=cx-c^2$ 之直線族的包絡線

4.對於Ⅱ中之任一點 p，至少存在一直線切於 $y=\frac{1}{4}x^2$ 且通過 p，即Ⅱ為有解之點集合。反之，對於Ⅰ中之任一點 Q，無法找出一條直線通過 Q 且切於 $y=\frac{1}{4}x^2$，所以Ⅰ為無解之點集合



壹階壹次常微分方程式

- 壹階壹次常微分方程式其表示方式有下列兩種型式

    (1) $y' = f(x, y)$

    (2) $M(x, y)dx + N(x, y)dy = 0$

- 題型分類

    (0) 直接積分 $y' = x$， $y = \frac{1}{2}x^2 + c$

    (1) 可<u>分離變數</u>

    (2) 壹階<u>正合</u>常微分方程式

    (3) 壹階<u>線性</u>常微分方程式

1.2 Separable Equations

- 可分離變數型之壹階壹次常微分方程式：

    (1) 可直接分離變數者

    (2) 可藉由變數變換轉為(1)者

- 若方程式給定為 $y' = f_1(x) f_2(y)$

$$\frac{dy}{f_2(y)} = f_1(x)dx$$

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + c$$

◆ 若方程式給定為$M_1(x)M_2(y)dx+N_1(x)N_2(y)dy=0$

$$\frac{M_1(x)}{N_1(x)}dx+\frac{N_2(y)}{M_2(y)}dy=0$$

$$\int\frac{M_1(x)}{N_1(x)}dx+\int\frac{N_2(y)}{M_2(y)}dy=c$$

Ex3： 求下列 O.D.E.之通解: $y'=y^2e^{-x}$

［解］：

Ex4(a): Solve the initial value problem $y'=y^2e^{-x}$; $y(1)=4$

［解］：

Ex4(b)：求下列 O.D.E.之通解：$(1+x^2)y'+2xy\ln y=0$

［解］：

Ex5: Estimate the time of death. The body is in a room that is kept at a constant 68°F. Body temperature is decreasing. Initial body temperature is 98.6°F. By observing the body’s current temperature, estimate the time of death. The rate of radiated heat energy into the room is proportional to the difference in temperature between the body and the room.

[解]:

Ex6: Radioactive decay and carbon dating. Mass is converted to energy by radiation. The change rate of mass is proportional to the mass itself.

[解]:

Ex7: A culture contains B_0 number of bacteria initially. At the time of 1 hour, the number of bacteria is $2B_0$. If the growth rate of bacteria is proportional to the number of bacteria N(t) presented at time t, determine the time required for the number of bacteria to $10B_0$.

[解]:

Exercise A:

1. $xy' + y = y^2$ $\{ y = cxy + 1 \}$

2. $2yy' = e^{x-y^2}; y(4) = -2$ $\{ y = -\sqrt{x} \}$

3. An object having a temperature of 90 degrees Fahrenheit is placed into an environment kept at 60 degrees. Ten minutes later, the object has cooled to 88 degrees. What will be the temperature of the object after it has been in this environment for 20 minutes? How long will it take for the object to cool to 65 degrees? $\{ T(20) = 60 + 30\left(\frac{14}{15}\right)^2 = 86.13$ °F, $t = \frac{10\ln(1/6)}{\ln(14/15)} \approx 259.7$ minutes$\}$

4. A radioactive element has a half-life of $\ln(2)$ weeks. If e^3 tons are present at a given time, how much will be left 3 weeks later? $\{ A(3) = e^3\left(\frac{1}{2}\right)^{3/\ln 2} = 1$ ton $\}$

5. Given that 12 grams of a radioactive element decays to 9.1 grams in 4 minutes, what is the half-life of this element? $\{ t = -\frac{\ln 2}{k} \approx 10.02$ minutes$\}$

◆ 可藉由變數變換轉成 Separable Equations

（1）壹階齊次 O.D.E.

（2）型式為$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$

（3）型式為$y'=f(ax+by+c)$

1.6 壹階齊次(Homogeneous) O.D.E.

定義：$y'=f(\frac{y}{x})$ （or $f(\frac{x}{y})$）為壹階齊次 O.D.E.

判斷方式:

(1) If $f(\lambda x,\lambda y)=f(x,y)$，則其為壹階齊次 O.D.E.

(2) If $M(\lambda x,\lambda y)=\lambda^m M(x,y)$，$N(\lambda x,\lambda y)=\lambda^m N(x,y)$，則其為壹階齊次 O.D.E.

Ex8：求解下列微分方程式：$(x-\sqrt{xy})y'=y$

［解］：

Ex9：求下列 O.D.E.之通解: $(3xy+y^2)+(x^2+xy)\dfrac{dy}{dx}=0$

［解］：

解法：

1.已知(判斷) $y' = f(x,y)$ 為壹階齊次常微分方程式

①原式可直接化為 $y' = f(\frac{y}{x})$

②令 $\frac{y}{x} = v$ 得 $y = vx$，且 $dy = vdx + xdv$

2. 已知(判斷) $M(x,y)dx + N(x,y)dy = 0$ 為壹階齊次常微分方程式

①將原式乘上 $\frac{1}{x^m}$ (或 $\frac{1}{y^m}$)化為：

$$M(\frac{y}{x})dx + N(\frac{y}{x})dy = 0$$

②令 $\frac{y}{x} = v$ 得 $y = vx$，且 $dy = vdx + xdv$

Ex10: Suppose a person jumps into a canal of constant width w and swims toward a fixed point directly opposite the point of entry into the canal. The person's speed is v and the water current's speed is s. Assume that on the way across, the swimmer always orients to point toward the target. We want to determine the swimmer's trajectory.

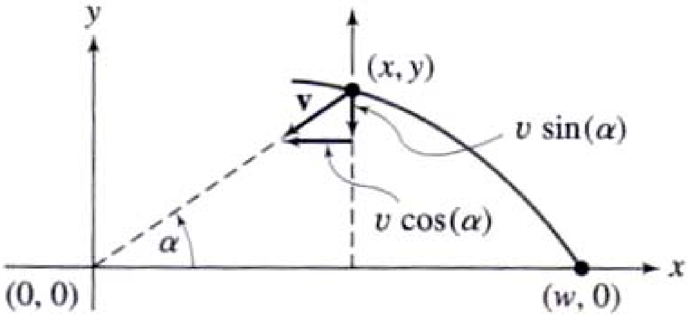


FIGURE 1.14 *The swimmer's path.*

[解]： $x'(t) = -v\cos(\alpha)$ and $y'(t) = s - v\sin(\alpha)$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{s - v\sin(\alpha)}{-v\cos(\alpha)} = \tan(\alpha) - \frac{s}{v}sec(\alpha)$$

$$\tan(\alpha) = \frac{y}{x} \text{ and } \sec(\alpha) = \frac{1}{x}\sqrt{x^2 + y^2}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\frac{1}{x}\sqrt{x^2 + y^2}, \quad \frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\sqrt{1 + \left(\frac{y}{x}\right)^2}$$

令 $y = ux$, $dy = udx + xdu$, $\dfrac{udx + xdu}{dx} = u - \dfrac{s}{v}\sqrt{1+u^2}$

$$u + x\frac{du}{dx} = u - \frac{s}{v}\sqrt{1+u^2}, \frac{1}{\sqrt{1+u^2}}du = -\frac{s}{v}\frac{1}{x}dx$$

$$\frac{d}{du}(\ln\left|u+\sqrt{1+u^2}\right|)=\frac{1}{u+\sqrt{1+u^2}}\left(1+\frac{1}{2}\frac{2u}{\sqrt{1+u^2}}\right)$$

$$1+\frac{u}{\sqrt{1+u^2}}=\frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}},$$

$$\frac{d}{du}(\ln\left|u+\sqrt{1+u^2}\right|)=\frac{1}{u+\sqrt{1+u^2}}\frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}}=\frac{1}{\sqrt{1+u^2}}$$

$$\ln\left|u+\sqrt{1+u^2}\right|=-\frac{s}{v}\ln|x|+C=\ln|x|^{\frac{-s}{v}}+\ln e^C=\ln\left(e^C|x|^{\frac{-s}{v}}\right)$$

$$u+\sqrt{1+u^2}=Kx^{-\frac{s}{v}}$$

$$\sqrt{1+u^2}=Kx^{-\frac{s}{v}}-u,\ \ 1+u^2=K^2x^{-\frac{2s}{v}}-2Kux^{-\frac{s}{v}}+u^2$$

$$u(x)=\frac{K^2x^{\frac{-2s}{v}}-1}{2Kx^{\frac{-s}{v}}},\ \ u(x)=\frac{1}{2}Kx^{-\frac{s}{v}}-\frac{1}{2}\frac{1}{K}x^{\frac{s}{v}}$$

$$y(x)=\frac{1}{2}Kx^{1-\frac{s}{v}}-\frac{1}{2}\frac{1}{K}x^{1+\frac{s}{v}},\ \ y(w)=0,$$

$$\frac{1}{2}Kw^{1-\frac{s}{v}}-\frac{1}{2}\frac{1}{K}w^{1+\frac{s}{v}}=0,\ \ Kw^{1-\frac{s}{v}}=\frac{1}{K}w^{1+\frac{s}{v}},\ \ K^2=w^{\frac{2s}{v}}$$

$$K=w^{\frac{s}{v}},\ \ y(x)=\frac{1}{2}w^{\frac{s}{v}}x^{1-\frac{s}{v}}-\frac{1}{2}w^{\frac{-s}{v}}x^{1+\frac{s}{v}}$$

$$y(x)=\frac{w}{2}w^{-1}w^{\frac{s}{v}}x^{1-\frac{s}{v}}-\frac{w}{2}w^{-1}w^{\frac{-s}{v}}x^{1+\frac{s}{v}},$$

$$y(x)=\frac{w}{2}\left[\left(\frac{x}{w}\right)^{1-\frac{s}{v}}-\left(\frac{x}{w}\right)^{1+\frac{s}{v}}\right]$$

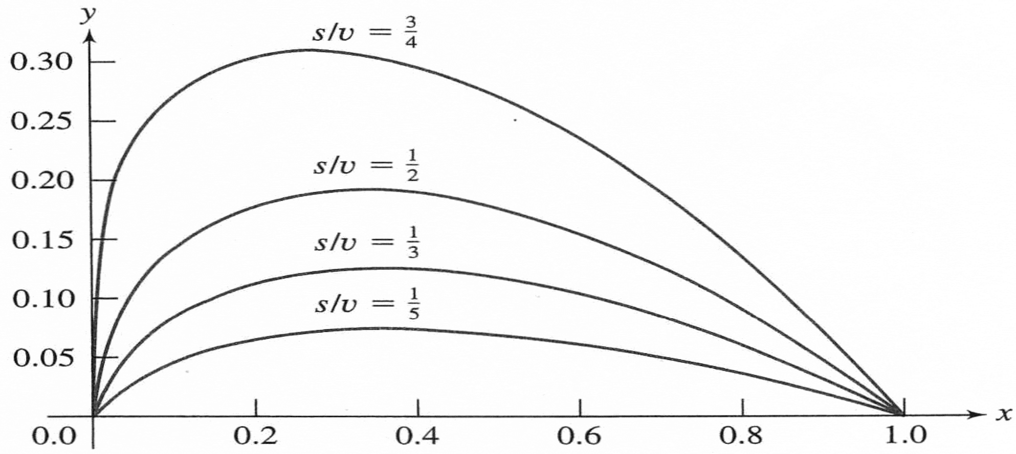


FIGURE 1.15 *Graphs of*

$$y = \frac{w}{2}\left[\left(\frac{x}{w}\right)^{1-s/v} - \left(\frac{x}{w}\right)^{1+s/v}\right]$$

for s/v *equal to* $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{2}$ *and* $\frac{3}{4}$, *and* w *chosen as* 1.

Exercise B:

1. $(2y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$ $\{x^5\left(\frac{y}{x}\right)^2\left(\frac{y}{x} - 2\right) = c\}$

2. $y' = \frac{1}{x^2}y^2 - \frac{1}{x}y + 1$ $\{y = x + \frac{x}{c - \ln(x)}\}$

3. $y' = \frac{y}{x + y}$ $\{y\ln(y) - x = cy\}$

4. A man stands at the junction of two perpendicular roads and his dog is watching him from one of the roads at a distance of A feet away. At a given instant the man starts to walk with constant speed v along the other road, and at the same time the dog begins to run toward the man with speed of $2v$. Determine the path the dog will take, assuming that it always moves so that it is facing the man. Also determine when the dog will eventually catch the man. $\{t = \frac{2A}{3v}\}$

◆ 型式為$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$

當$c_1=c_2=0$時，其為壹階齊次常微分方程式。If $c_1\neq 0$，or $c_2\neq 0$，再分成兩類:

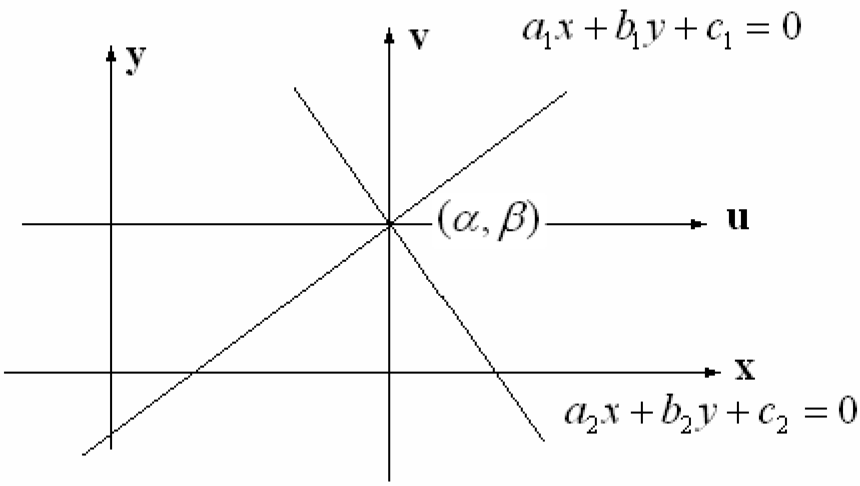
◆ 若$\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$，則兩直線相交於一點

$$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases}$$

之解$(x,y)=(\alpha,\beta)$ 如下圖

令$\begin{cases} x=u+\alpha \\ y=v+\beta \end{cases}$ 得 $\begin{cases} dx=du \\ dy=dv \end{cases}$

$(a_1u+b_1v)du+(a_2u+b_2v)dv=0$ 為壹階齊次常微分方程式



Ex11：求下列常微分方程式之通解：

$$(-3x+y+6)dx+(x+y+2)dy=0$$

[解]：

◆ 若 $\frac{a_1}{a_2}=\frac{b_1}{b_2}=m\neq\frac{c_1}{c_2}$

令 $a_2x+b_2y=z \qquad a_2dx+b_2dy=dz$

$a_1x+b_1y=mz \qquad dy=\frac{dz-a_2dx}{b_2}$

$$(mz+c_1)dx+(z+c_2)\frac{dz-a_2dx}{b_2}=0$$

$$[b_2(mz+c_1)-a_2(z+c_2)]dx+(z+c_2)dz=0$$

$$dx+\frac{(z+c_2)dz}{(b_2m-a_2)z+(b_2c_1-a_2c_2)}=0$$

再積分

Ex12：求下列 O.D.E.之通解:

$$(x+y)dx+(3x+3y-4)dy=0$$

［解］：

Exercise C:

1. $y' = \dfrac{(x-y+2)^2}{(x+1)^2}$

$$\left\{ \frac{y-\alpha x-(1+\alpha)}{y-\beta x-(1+\beta)} = c\left(x+1\right)^{\sqrt{5}}, \alpha = \frac{3+\sqrt{5}}{2}, \beta = \frac{3-\sqrt{5}}{2} \right\}$$

◆ 型式為 $y' = f(ax+by+c)$

$$t = ax+by+c$$

$$dy = \frac{dt - adx}{b}$$

Ex13：求 O.D.E. $y' = \tan^2(x+y)$ 之通解

[解]：

Exercise D:

1. $(x+y)y'=1$ $\{\ln|x+y+1|=y+c\}$

2. $y'=y^2-2xy+x^2+1$ $\{\ x+\dfrac{1}{y-x}=c\}$

3. $y'-2xy=x^2+y^2$ $\{\ y=\tan(x+c)-x\}$

1.4 Exact Differential Equations

(1) 壹階正合(exact) O.D.E.

(2) 可正合化之壹階壹次 O.D.E

◆ 定義：

考慮壹階壹次常微分方程式為

$M(x,y)dx+N(x,y)dy=0$。若存在有一函數$\phi(x,y)$能夠使

得$d\phi=\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy=M(x,y)dx+N(x,y)dy$

即
$$\begin{cases}\dfrac{\partial\phi}{\partial x}=M(x,y)\\ \dfrac{\partial\phi}{\partial y}=N(x,y)\end{cases}$$

則稱 $M(x,y)dx+N(x,y)dy=0$ 為正合方程式 (Exact Equation)。其中，$\phi(x,y)=c$ 即為此正合 O.D.E.之通解

◆ 壹階正合 O.D.E.之判別式：

由定義知，若$M(x,y)dx+N(x,y)dy=0$為正合時，則存在一函數$\phi(x,y)$使得

$$\frac{\partial \phi}{\partial x}=M(x,y) \quad \text{且} \quad \frac{\partial \phi}{\partial y}=N(x,y)$$

今假設$\phi(x,y)$於給定區域當中，具有連續二階偏導數，則由

$$\frac{\partial M}{\partial y}=\frac{\partial^2 \phi}{\partial y \partial x}=\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial N}{\partial x}$$

得出$M(x,y)dx+N(x,y)dy=0$為正合時之判別式為

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Note

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$為$M(x,y)dx+N(x,y)dy=0$為正合之充分必要條件

◆ 壹階正合 O.D.E.之求解

設壹階 O.D.E. $M(x,y)dx + N(x,y)dy = 0$ 為正合時，(即 $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)，則可令原式之通解為 $\phi(x,y) = c$

且 $$\begin{cases} \frac{\partial \phi}{\partial x} = M(x,y) \\ \frac{\partial \phi}{\partial y} = N(x,y) \end{cases}$$

先將上兩式做偏積分可得

$$\phi(x,y) = \int^x M(x,y)dx + f(y)$$

$$\phi(x,y) = \int^y N(x,y)dy + g(x)$$

再比較上兩式得出 $f(y)$ 及 $g(x)$，即可求出 $\phi(x,y)$

Ex14：試求解壹階微分方程式：

$$(3x^2y^2+e^y)\frac{dy}{dx}+2(xy^3+1)=0$$

[解]：

Ex15：求下列 O.D.E.之通解：

$$(x^2-4xy-y^2)dx+(y^2-2xy-2x^2)dy=0$$

[解]：

Exercise E:

1. $\frac{1}{x} + y + (3y^2 + x)y' = 0 \quad \{\ln\left(x\right) + xy + y^3 = c\}$

1.5 Integrating Factors

- 可正合化之壹階壹次 O.D.E.

  考慮壹階壹次 O.D.E. $M(x,y)dx+N(x,y)dy=0$

  若上式中之 $\dfrac{\partial \mathrm{M}}{\partial \mathrm{y}} \neq \dfrac{\partial \mathrm{N}}{\partial \mathrm{x}}$，則由前述知，此時必存在一函數 $I(x,y)$ 使得 $I(x,y)M(x,y)dx+I(x,y)N(x,y)dy=0$ 為正合，即 $\dfrac{\partial(IM)}{\partial y}=\dfrac{\partial(IN)}{\partial x}$。我們稱此 $I(x,y)$ 為所考慮之非正合 O.D.E.的積分因子（Integrating factor）

- 就理論上而言，任何壹個壹階非正合 O.D.E.，均至少存在一個積分因子，可使其化為正合。但是，我們必須了解的是，並非任何壹階非正合 O.D.E. ，其積分因子均可很容易求得；所以，如何快速求得壹階非正合 O.D.E.之積分因子，即為此章節所研討之主題

◆ 積分因子$I(x,y)$之求法：

由前述知$I(x,y)$滿足 $\dfrac{\partial(IM)}{\partial y}=\dfrac{\partial(IN)}{\partial x}$

將上式展開得

$$I(x,y)\frac{\partial M}{\partial y}+M(x,y)\frac{\partial I}{\partial y}=I(x,y)\frac{\partial N}{\partial x}+N(x,y)\frac{\partial I}{\partial x}$$

$$N(x,y)\frac{\partial I}{\partial x}-M(x,y)\frac{\partial I}{\partial y}=(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})I(x,y)$$

1. $I(x,y)=I(x)$：若已知$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}dx=\frac{dI}{I},\ 即\ f(x)dx=\frac{dI}{I}$$

∴原式具有一積分因子為：

$$I(x)=\mathrm{e}^{\int f(x)dx}$$

2. $I(x,y)=I(y)$ ：若已知 $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}=f(y)$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}dy=\frac{dI}{I},\text{ 即 } f(y)dy=\frac{dI}{I}$$

∴原式具有一積分因子為：

$$I(y)=\mathrm{e}^{\int f(y)dy}$$

3. $I(x,y)=I(x+y)$ ：若已知 $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N-M}=f(x+y)$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N-M}du=\frac{dI}{I},\ u=x+y\text{ ，即 } f(u)du=\frac{dI}{I}$$

∴原式具有一積分因子為：

$$I(u)=\mathrm{e}^{\int f(u)du}$$

4. $I(x,y)=I(xy)$：若已知$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{Ny-Mx}=f(xy)$

$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{Ny-Mx}du=\dfrac{dI}{I}$, $u=xy$，即 $f(u)du=\dfrac{dI}{I}$

∴原式具有一積分因子為：

$$I(u)=\mathrm{e}^{\int f(u)du}$$

(表一)

檢驗條件	積分因子
$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$	$I(x)=\mathrm{e}^{\int f(x)dx}$
$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}=f(y)$	$I(y)=\mathrm{e}^{\int f(y)dy}$
$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N-M}=f(x+y)$	$I(x+y)=\mathrm{e}^{\int f(x+y)d(x+y)}$
$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{Ny-Mx}=f(xy)$	$I(xy)=\mathrm{e}^{\int f(xy)d(xy)}$

Ex16：Solve the differential equation：

$$\left(4x+3y^2\right)dx+2xydy=0$$

［解］：

Ex17：求下列 O.D.E. 之通解

$$\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$$

[解]:

Exercise F:

1. $(3xy+y^2)+(x^2+xy)\dfrac{dy}{dx}=0$ $\{x^3y+\frac{1}{2}x^2y^2=c\}$

2. $2xydx+(4y+3x^2)dy=0;\, y(1)=1$ $\{y^4+x^2y^3=2\}$

1.3 Linear Differential Equations

(1) 壹階線性 O.D.E.

(2) 可線性化之壹階壹次 O.D.E.

◆ 壹階線性常微分方程式：

壹階線性常微分方程式 $y'+P(x)y=Q(x)$

解法：積分因子法

1. 因 $y'+P(x)y=Q(x)$ 可以改寫為

$\left[P(x)y-Q(x)\right]dx+dy=0\cdots\cdots(1)$

由(1)式知，$\dfrac{\partial M}{\partial y}=\dfrac{\partial}{\partial y}\left[P(x)y-Q(x)\right]=P(x)$

且 $\dfrac{\partial N}{\partial x}=\dfrac{\partial 1}{\partial x}=0$，故 $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$，(1)式不為正合 O.D.E.

2. $\because \dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=P(x)$

知(1)式具有一積分因子為：$I(x)=e^{\int P(x)dx}$

Note

$$I(x)\left[P(x)y-Q(x)\right]dx+I(x)dy=0$$

$$I(x)P(x)ydx+I(x)dy=I(x)Q(x)dx$$

$$I(x)\frac{dy}{dx}+P(x)I(x)y=I(x)Q(x)$$

$$e^{\int P(x)dx}y'(x)+P(x)e^{\int P(x)dx}y(x)=Q(x)e^{\int P(x)dx}$$

$$\frac{d}{dx}\left(y(x)e^{\int P(x)dx}\right)=Q(x)e^{\int P(x)dx}\quad \text{(always)}$$

$$y(x)e^{\int P(x)dx}=\int\left(Q(x)e^{\int P(x)dx}\right)dx+c$$

$$y(x)=e^{-\int P(x)dx}\int\left(Q(x)e^{\int P(x)dx}\right)dx+ce^{-\int P(x)dx}$$

Ex18 : Find a general solution of $y'+y=\sin(x)$

［解］：

Ex19：Find a general solution of $y' = \dfrac{y}{2x + y^3 e^y}$

［解］：

Exercise G:

1. $2y' + 3y = e^{2x}$ $\{y = \frac{1}{7}e^{2x} + ce^{-3x/2}\}$

2. $y' + \frac{4}{x}y = 2; y(1) = -4$ $\{y = \frac{2}{5}x - \frac{22}{5}x^{-4}\}$

3. $y' = 1 + \cos x - y\cot x; y(\frac{\pi}{6}) = \frac{1}{4}$ $\{y = \frac{\frac{\sqrt{3}}{2}}{\sin x} - \cot x + \frac{1}{2}\sin x\}$

1.6 Bernoulli, Riccati Equations

◆ 可線性化之壹階壹次 O.D.E.

區分成三類:

(1)Bernoulli 方程式: $y'+P(x)y=Q(x)y^n$

(2) $\dfrac{dv}{dy}y'+P(x)v(y)=Q(x)$

(3)Riccati 方程式: $y'=P(x)y^2+Q(x)y+R(x)$

◆ Bernoulli 方程式：

壹階壹次常微分方程式其型式為：

$$y'+P(x)y=Q(x)y^n$$

稱為 Bernoulli 方程式。(其中 $n\neq 0,1$)

Step1：

將式子改寫為：

$$y^{-n}y'+P(x)y^{1-n}=Q(x)$$

Step2：

令 $y^{1-n}=u$， 得 $\dfrac{du}{dx}=(1-n)y^{-n}y'$

$\dfrac{1}{1-n}\dfrac{du}{dx}+P(x)u=Q(x)$ 此為壹階線性 O.D.E.

Ex20：求下列 O.D.E.之通解：$dy+2xydx=xe^{-x^2}y^3dx$

［解］：

Exercise H:

1. $y'+\frac{1}{x}y=x^3y^3 \quad \{\frac{1}{y^2}=-x^4+cx^2\}$

2. $xy'=(y-x)^3+y \quad \{\frac{1}{(y-x)^2}=\frac{c}{x^2}-1\}$

3. $x^3y'=x^2y-y^3 \quad \{\frac{1}{2}\frac{x^2}{y^2}=\ln|x|+c\}$

4. $y'+\frac{2}{x}y=\frac{3}{x}y^2 \quad \{y=\frac{2}{3+cx^2}\}$

◆ $\frac{dv}{dy}y'+P(x)v(y)=Q(x)$

令 $v(y)=u$ ，$\frac{du}{dx}=\frac{dv}{dy}y'$ 代入

得 $\frac{du}{dx}+P(x)u=Q(x)$ ，此為壹階線性 O.D.E.

Ex21：求解下列 O.D.E.之通解：$e^{y}y'=3\left(x+e^{y}\right)-1$

[解]：

Exercise I:

1. $y'\sin y + \sin x \cos y = \sin x$ $\{-\ln\left|\cos y - 1\right| - \cos x = c\}$

◆ Riccati 方程式：

壹階壹次 O.D.E.型式為：

$y' = P(x)y^2 + Q(x)y + R(x)$ ，稱為 Riccati 方程式

若 $P(x) = 0$，則為壹階線性 O.D.E.

◆ Riccati 方程式之求解:

通常必須先得到一個特解 $y = S(x)$，$S(x)$ 通常是經由觀察，

猜測及嘗試錯誤等方式求得

通解 $y = S(x) + \dfrac{1}{z(x)}$（$z(x)$ 為待定函數）

$$y' = S'(x) + \frac{-z'(x)}{z^2(x)}$$

Ex22：求解下列 O.D.E.之通解：$y'=\frac{1}{x}y^2+\frac{1}{x}y-\frac{2}{x}$

［解］：

Exercise J:

1. $y' = \frac{1}{2x}y^2 - \frac{1}{x}y - \frac{4}{x} \quad \{ y = 4 + \frac{6x^3}{c - x^3} \}$

2. $y' = -e^{-x}y^2 + y + e^x \quad \{ y = \frac{2ce^{3x} + e^x}{2ce^{2x} - 1} \}$

1.7 Applications

◆ Charge $q(t)$ and current $i(t)$ => $i(t)=q'(t)$

Voltage drop => resistor R is iR

=> capacitor C is $\frac{q}{C}$

=> inductor L is $Li'(t)$

Ex23(a): Find the current in the RL circuit

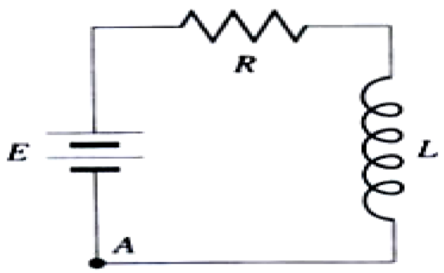


FIGURE 1.18 *RL Circuit.*

[解]:

Ex23(b): Find the current in the RC circuit

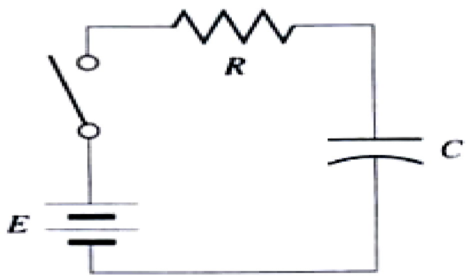
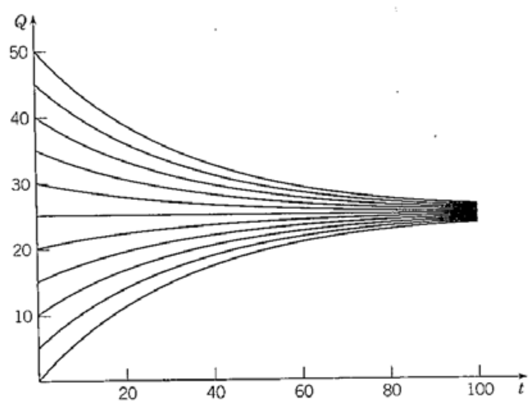


FIGURE 1.19 *RC circuit.*

[解]:

Ex24: A container has Q_0g of sugar dissolved in 100 liters of water. Assume the water containing 0.25g of water per liter is flew into the container at a rate of r liters per minute and the well-stirred sugar water is draining from the container at the same rate. Determine the quantity of sugar Q(t) in the container at any time. Also find the limiting quantity QL that is present after a very long time.

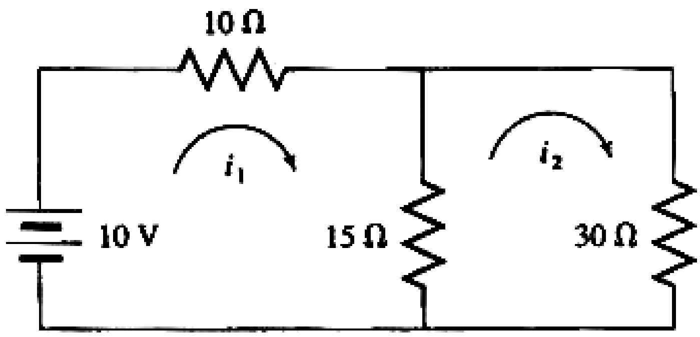
[解]:



Exercise K:

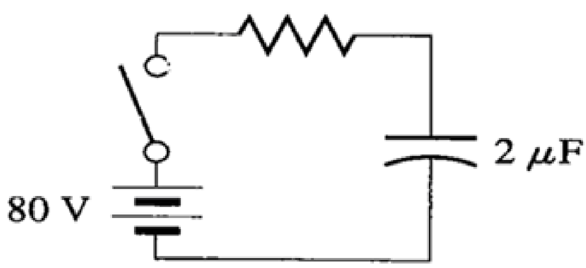
1. Determine each of the currents in the following circuit.

$\{i_1 = \frac{1}{2}A, i_2 = \frac{1}{6}A\}$



2. In the following circuit, the capacitor is initially discharged and R is 250kΩ. How long after the switch is closed will the capacitor voltage be 76 volts? Determine the current in the resistor at the time. (Here μF denotes 10^{-6} farads.)

$\{t = 0.5\ln(20) = 1.498 sec,\ \ i = 16\mu A\}$



3. Suppose a person carrying a virus returns to an isolated group of 100 persons. Assume that the day rate at which the virus spreads out is proportional to the multiplication of the number of infected persons and the number of non-infected persons. It is observed that the number of infected persons is 7 after 4 days. Determine the number of infected persons after 10 days. {60 persons after truncation}

壹階高次微分方程式：

$F(x,y,y')=0$，若其 y' 之次數大於 1 次，為壹階高次微分方程式

$$a_n(x,y)p^n+....+a_1(x,y)p+a_0(x,y)=0$$

其中，$p\equiv y'$，$a_n(x,y)\neq 0$

◆ 分解因式法：

$$(p-f_1(x,y))(p-f_2(x,y))......(p-f_n(x,y))=0$$

可得：
$$\begin{cases} p=y'=f_1(x,y)....................(1) \\ p=y'=f_2(x,y)....................(2) \\ \vdots \\ p=y'=f_n(x,y)....................(n) \end{cases}$$

因 (1) 至 (n) 均為壹階壹次 O.D.E.

$$\begin{cases} \phi_1(x,y)=c \\ \phi_2(x,y)=c \\ \vdots \\ \phi_n(x,y)=c \end{cases}$$

$$\left[\phi_1(x,y)-c\right]\left[\phi_2(x,y)-c\right]...\left[\phi_n(x,y)-c\right]=0$$

Ex25：求下列 O.D.E.之通解：$\left(y'\right)^3 - y' = 0$

［解］：

Ex26：試求解下列方程式之通解

$$xyp^2 + \left(x^2 + xy + y^2\right)p + x\left(x + y\right) = 0,\ p \equiv y'$$

［解］：

常數與指數的積分

$\int dx = x + C$	$\int adx = a\int dx = ax + C$
$\int x^r dx = \frac{x^{r+1}}{r+1} + C$	$\int \frac{dx}{x} = \ln\|x\| + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C, a > 0$

三角函數的積分

$\int \sin xdx = -\cos x + C$	$\int \cos xdx = \sin x + C$
$\int \tan dx = \ln\|\sec x\| + C$	$\int \cot xdx = \ln\|\sin x\| + C$
$\int \sec dx = \ln\|\sec x + \tan x\| + C$	$\int \csc dx = \ln\|\csc x - \cot x\| + C$
$\int \sec^2 xdx = \tan x + C$	$\int \csc^2 xdx = -\cot x + C$
$\int \sec x \tan xdx = \sec x + \mathrm{C}$	$\int \csc x \cot xdx = -\csc x + C$

反三角函數的積分

$\int \sin^{-1} xdx = x\sin^{-1} x + \sqrt{1-x^2} + C$
$\int \cos^{-1} xdx = x\cos^{-1} x - \sqrt{1-x^2} + C$
$\int \tan^{-1} xdx = x\tan^{-1} x - \ln\sqrt{1-x^2} + C$
$\int \cot^{-1} xdx = x\cot^{-1} x + \ln\sqrt{1-x^2} + C$
$\int \sec^{-1} xdx = x\sec^{-1} x - \ln\|x + \sqrt{x^2-1}\| + C$
$\int \csc^{-1} xdx = x\csc^{-1} x + \ln\|x + \sqrt{x^2-1}\| + C$

畢氏定理

$\sin^2 x+\cos^2 x=1$	$\tan^2 x+1=\sec^2 x$	$1+\cot^2 x=\csc^2 x$

符號等式

$\sin(-x)=-\sin x$	$\cos(-x)=\cos x$	$\tan(-x)=-\tan x$
$\csc(-x)=-\csc x$	$\sec(-x)=\sec x$	$\cot(-x)=-\cot x$

餘角等式

$\sin\left(\frac{\pi}{2}-x\right)=\cos x$	$\cos\left(\frac{\pi}{2}-x\right)=\sin x$	$\tan\left(\frac{\pi}{2}-x\right)=\cot x$
$\csc\left(\frac{\pi}{2}-x\right)=\sec x$	$\sec\left(\frac{\pi}{2}-x\right)=\csc x$	$\cot\left(\frac{\pi}{2}-x\right)=\tan x$

補角等式

$\sin(\pi-x)=\sin x$	$\cos(\pi-x)=-\cos x$	$\tan(\pi-x)=-\tan x$
$\csc(\pi-x)=\csc x$	$\sec(\pi-x)=-\sec x$	$\cot(\pi-x)=-\cot x$
$\sin(\pi+x)=-\sin x$	$\cos(\pi+x)=-\cos x$	$\tan(\pi+x)=\tan x$
$\csc(\pi+x)=-\csc x$	$\sec(\pi+x)=-\sec x$	$\cot(\pi+x)=\cot x$

複角公式

$\sin(x+y)=\sin x\cos y+\cos x\sin y$
$\sin(x-y)=\sin x\cos y-\cos x\sin y$
$\cos(x+y)=\cos x\cos y-\sin x\sin y$
$\cos(x-y)=\cos x\cos y+\sin x\sin y$
$\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$
$\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

倍角公式

$\sin 2x=2\sin x\cos x$	$\cos 2x=\cos^2 x-\sin^2 x$
$\cos 2x=2\cos^2 x-1$	$\cos 2x=1-2\sin^2 x$

半角公式

$\sin^2\dfrac{x}{2}=\dfrac{1-\cos x}{2}$	$\cos^2\dfrac{x}{2}=\dfrac{1+\cos x}{2}$