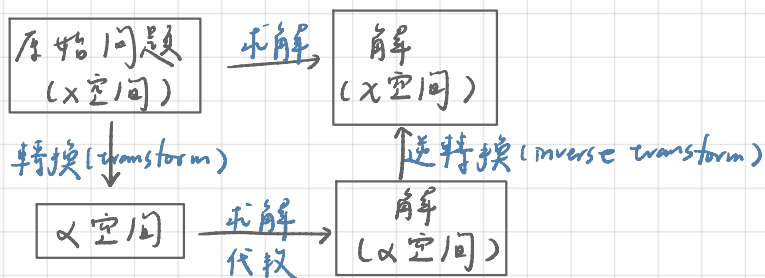


The Laplace Transform

積分轉換 (Integral transform)

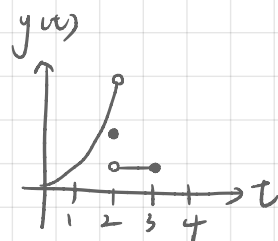
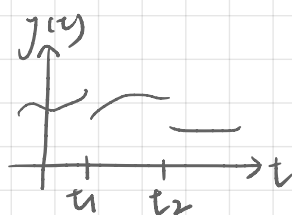
$f(x)$ 乘上 核函數 (kernel) $k(\alpha, x)$, $x \in (a, b)$, 積分
 $\int_a^b f(x) k(\alpha, x) dx = F(\alpha) \quad f(x) \leftrightarrow F(\alpha)$



分段連續函數 (piecewise continuous function)

$$f(t) \quad a \leq t \leq b$$

1. 不連續點為有限個
2. 若令不連續點, 左右極限存在



指數階層函數 (exponential order function)

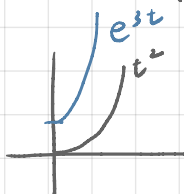
存在 $M > 0, \alpha, T$

$$|f(t)| < M \cdot e^{\alpha t}, \quad t > T$$

$t \rightarrow \infty$, $f(t)$ 為 α 指數階層函數

example: $f(t) = t^2$

$$|f(t)| < \frac{1}{M} e^{3t}$$



example: $f(t) = e^{t^2}$

$|f(t) \cdot e^{-\alpha t}| \not< M \Rightarrow$ 不是 α 指數階層函數

Laplace 轉換之定義

$f(t), t > 0$

$$\mathcal{L}[f(t)] = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) \boxed{e^{-st}} dt = F(s) \quad f(t) \xrightarrow{\mathcal{L}} F(s)$$

kernel function

Laplace 轉換存在的充分條件 (積分值收斂)

1. $T > 0^+$ $f(t)$ 於 $t \in [0^+, T]$ 上為分段連續

2. α 指數階層函數 $\forall s > \alpha$ $\mathcal{L}[f(t)]$ 存在

$$\int_0^\infty |f(t)| e^{-st} dt < \int_0^\infty M e^{\alpha t} e^{-st} dt = M \int_0^\infty e^{(\alpha-s)t} dt = \frac{M}{(\alpha-s)} e^{(\alpha-s)t} \Big|_0^\infty$$

$$= \frac{M}{\alpha-s} (0-1) = \frac{-M}{\alpha-s}$$

$s > \alpha$

$\uparrow$

要收斂 $(\alpha-s)t$ 要小於 0

基本函數之 Laplace 轉換

1. $\mathcal{L}\{t^a\} = \int_0^\infty t^a e^{-st} dt \quad (a=0, 1, 2, 3, \dots)$

令 $z = st, t = \frac{z}{s}, dt = \frac{dz}{s}$

$$= \int_0^\infty \left(\frac{z}{s}\right)^a e^{-z} \frac{dz}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty z^a e^{-z} dz$$

$$= \frac{1}{s^{a+1}} \int_0^\infty z^{(a+1)-1} e^{-z} dz = \frac{a!}{s^{a+1}}$$

Gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\Gamma(a+1) = a!$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n\Gamma[(n-1)+1]$$

$$= n(n-1)\Gamma(n-1)$$

$$= \dots \Gamma[(n-2)+1]$$

$$= \dots \Gamma(1) = 1$$

$$= n!$$

$$\Gamma(n+1) = n\Gamma(n)$$

pf: $\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x} dx$

$$= -x^n e^{-x} \Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx$$

$$= \frac{-x^n}{e^x} \Big|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$= \int_0^\infty x^n e^{-x} dx$$

$$= 0 + n\Gamma(n)$$

令 $u = x^n, dv = e^{-x} dx$

$du = nx^{n-1}, v = -e^{-x}$

$$\frac{-x^n}{e^x} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopital}} \frac{-nx^{n-1}}{e^x} = \dots = 0$$

L'Hopital

2. $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st} dt$

$$= \int_0^\infty e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty = \frac{0-1}{a-s} = \frac{1}{s-a}$$

3. $\mathcal{L}\{\cos at\} = ? \quad \mathcal{L}\{\sin at\} = ?$

Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat} dt$$

$$= \int_0^\infty e^{-st} (\cos at + i \sin at) dt$$

$$= \int_0^\infty \cos at \cdot e^{-st} dt + i \int_0^\infty \sin at \cdot e^{-st} dt$$

$$= \mathcal{L}\{\cos at\} + i \mathcal{L}\{\sin at\}$$

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$$

$$= \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2}$$

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

逆轉換 (inverse)

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^{a+1}}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{a!}{s^{a+1}}\right\} = \frac{1}{a!} \mathcal{L}^{-1}\left\{\frac{a!}{s^{a+1}}\right\} = \frac{t^a}{a!}$$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2a}\right\} = e^{2at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2s-a}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-\frac{a}{2}}\right\} = \frac{1}{2} e^{\frac{a}{2}t}$$

3. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \mathcal{L}^{-1}\left\{\frac{a}{a(s^2+a^2)}\right\} = \frac{1}{a} \sin at$$

Laplace transformation 之基本定理

線性運算

設 $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$

$$\mathcal{L}\{C_1 f(t) + C_2 g(t)\} = C_1 F(s) + C_2 G(s)$$

pf: $\mathcal{L}\{C_1 f(t) + C_2 g(t)\} = \int_0^\infty [C_1 f(t) + C_2 g(t)] e^{-st} dt$
 $= C_1 \int_0^\infty f(t) e^{-st} dt + C_2 \int_0^\infty g(t) e^{-st} dt = C_1 F(s) + C_2 G(s)$

第一平移定理 Theorem and Heaviside Function

設 $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{f(t) e^{at}\} = F(s-a)$$

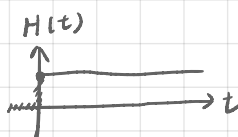
pf: $\mathcal{L}\{f(t) e^{at}\} = \int_0^\infty f(t) e^{at} \cdot e^{-st} dt$
 $= \int_0^\infty f(t) e^{-(s-a)t} dt = F(s-a)$

example: $\mathcal{L}\{e^{at} \cos bt\} = \frac{(s-a)}{(s-a)^2 + b^2}$ 所有 s 的地方都換成 $(s-a)$

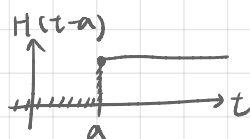
推理: $\mathcal{L}^{-1}\{F(s)\} = f(t)$
 $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



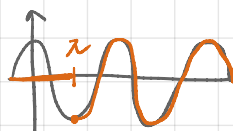
$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



$$H(t-a) \cdot g(t) = \begin{cases} 0 & \text{if } t < a \\ g(t) & \text{if } t \geq a \end{cases}$$

截掉左邊

example: $H(t-\pi) g(t) = \begin{cases} 0 & \text{if } t < \pi \\ \cos t & \text{if } t \geq \pi \end{cases}$



$$H(t-a)g(t-a) = \begin{cases} 0 & \text{if } t < a \\ g(t-a) & \text{if } t \geq a \end{cases} \quad \text{shift}$$

$$\text{example } H(t-\lambda)g(t-\lambda) = \begin{cases} 0 & \text{if } t < \lambda \\ \cos t & \text{if } t \geq \lambda \end{cases}$$

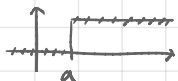


$$H(t) = u(t)$$

$$\mathcal{L}\{u(t)\} = \int_0^{\infty} u(t) \cdot e^{-st} dt = \int_0^{\infty} 1 \cdot e^{-st} dt = \frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

Pulse 脈波

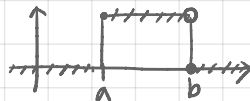
$$H(t-a)$$



$$H(t-b)$$

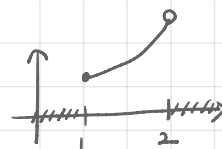


$$H(t-a) - H(t-b) \Rightarrow$$



example:

$$[H(t-1) - H(t-2)]e^t = \begin{cases} 0 & \text{if } t < 1 \\ e^t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$



第二平移定理

$$\text{設 } \mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{f(t-a) \cdot u(t-a)\} = e^{-as} F(s)$$

$$f(t-a)u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

$$\text{pf: } \mathcal{L}\{f(t-a)u(t-a)\} = \int_0^{\infty} f(t-a)u(t-a) \cdot e^{-st} dt$$

$$= \int_a^{\infty} f(t-a) e^{-st} dt$$

$$\downarrow \text{令 } \tau = t-a, t = \tau+a, dt = d\tau$$

$$= \int_0^{\infty} f(\tau) e^{-s(\tau+a)} d\tau$$

$$= e^{-as} \int_0^{\infty} f(\tau) e^{-s\tau} d\tau = e^{-as} F(s)$$

example:

$$\mathcal{L}\{(t+1)u(t-1)\}$$

$$\langle \text{誤解} \rangle e^{-1s} \cdot \mathcal{L}\{t+1\} = e^{-s} \left(\frac{2}{s^3} + \frac{1}{s} \right)$$

<正解>

$$(t+1) = (t-1) + 2t$$

$$= (t-1) + 2(t-1) + 2 \Rightarrow f(t-1)$$

$$f(t) = t+1$$

$$= g(t-1)$$

$$\mathcal{L}\{f(t) \cdot u(t-1)\} = \mathcal{L}\{[(t-1) + 2(t-1) + 2] \cdot u(t-1)\}$$

$$= e^{-s} \cdot G(s)$$

$$= e^{-s} \cdot \mathcal{L}\{t^2 + 2t + 2\} = e^{-s} \left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s} \right]$$

推理: $\mathcal{L}\{f(t) \cdot u(t-a)\}$

$$= \mathcal{L}\{g(t-a) \cdot u(t-a)\} = e^{-as} G(s) = e^{-as} \mathcal{L}\{g(t)\} = e^{-as} \mathcal{L}\{f(t+a)\}$$

$$f(t) = g(t-a), \quad \begin{matrix} \text{令 } t-a = \tau \\ t = \tau+a \end{matrix} \quad \begin{matrix} g(\tau) = f(\tau+a) \\ g(t) = f(t+a) \end{matrix}$$

$$\mathcal{L}\{(t+1) \cdot u(t-1)\} = e^{-1 \cdot s} \mathcal{L}\{f(t+1)\} = e^{-s} \mathcal{L}\{(t+1)+1\} = e^{-s} \mathcal{L}\{t^2+2t+2\}$$

example $f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ e^x & 1 \leq x < 4 \\ 0 & x \geq 4 \end{cases}$

$$f(x) = \underbrace{[u(x) - u(x-1)]}_{0 \sim 1} + e^x \underbrace{[u(x-1) - u(x-4)]}_{1 \sim 4} + 0$$

$$= u(x) + (e^x - 1) \cdot u(x-1) - e^x u(x-4)$$

$$\mathcal{L}\{f(x)\} = \frac{1}{s} + e^{-s} \mathcal{L}\{e^{(x+1)} - 1\} - e^{-4s} \mathcal{L}\{e^{(x+4)}\}$$

$$= \frac{1}{s} + e^{-s} [e^1 \cdot \frac{1}{s-1} - \frac{1}{s}] - e^{-4s} [e^4 \cdot \frac{1}{s-1}]$$

example $f(t) e^{-3t}$

$$f(t) = \begin{cases} 0 & t < 8 \\ t^2 - 4 & t \geq 8 \end{cases}$$

<解>

$$f(t) = (t^2 - 4) \cdot u(t-8)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t^2 - 4) u(t-8)\} = e^{-8s} \mathcal{L}\{f(t+8)\} = e^{-8s} \mathcal{L}\{(t+8)^2 - 4\}$$

$$= e^{-8s} \mathcal{L}\{t^2 + 16t + 60\} = e^{-8s} \left[\frac{2}{s^3} + \frac{16}{s^2} + \frac{60}{s} \right]$$

$$\mathcal{L}\{f(t) e^{-3t}\} = e^{-8(s+3)} \left(\frac{2}{(s+3)^3} + \frac{16}{(s+3)^2} + \frac{60}{s+3} \right)$$

尺度变换

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad a > 0$$

pf: $\mathcal{L}\{f(at)\} \quad \text{令 } at = \tau, \tau = \frac{\tau}{a}, d\tau = \frac{d\tau}{a}$

$$= \int_0^\infty f(at) \cdot e^{-st} d\tau \Rightarrow \int_0^\infty f(\tau) e^{-s(\frac{\tau}{a})} \frac{d\tau}{a} = \frac{1}{a} \int_0^\infty f(\tau) e^{-\left(\frac{s}{a}\right)\tau} d\tau = \frac{1}{a} F\left(\frac{s}{a}\right)$$

example $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2 + 3^2}$

$$\mathcal{L}\{\sin \underbrace{t}_a\} = \frac{1}{s^2 + 1} \quad \mathcal{L}\{\sin \underbrace{3t}_a\} = \frac{1}{3} F\left(\frac{s}{3}\right) = \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 + 1} = \frac{3}{s^2 + 9}$$

積分的 Laplace 轉換

設 $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s)$$

pf: $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}$

$$= \int_0^\infty \underbrace{\left[\int_0^t f(\tau) d\tau\right]}_u \underbrace{e^{-st}}_v dt \quad \frac{du}{dt} = f(t), v = -\frac{1}{s} e^{-st}$$

$$= \frac{-e^{-st}}{s} \left[\int_0^t f(\tau) d\tau \right] \Big|_0^\infty - \int_0^\infty \frac{-e^{-st}}{s} f(t) dt$$

$$= \underbrace{\left(\frac{\int_0^t f(\tau) d\tau}{-s e^{st}} \Big|_{t=\infty} - \frac{\int_0^t f(\tau) d\tau}{-s e^{st}} \Big|_{t=0} \right)}_{=0} + \frac{1}{s} \int_0^\infty f(t) e^{-st} dt = \frac{1}{s} F(s)$$

example: $\mathcal{L}\left\{\int_0^t \sin x dx\right\}$

$$\mathcal{L}\{\sin t\} = \frac{2}{s^2 + 4}$$

$$\mathcal{L}\left\{\int_0^t \sin x dx\right\} = \frac{1}{s} \cdot \frac{2}{s^2 + 4}$$

推論 $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s)$

$$\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\}$$

example $\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2 + 1}\right\} = \int_0^t \sin \tau d\tau = -\cos \tau \Big|_0^t = -\cos t + 1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2 + 1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2 + 1}\right\} = 1 - \cos t$$

Laplace 轉換的積分

設 $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du$$

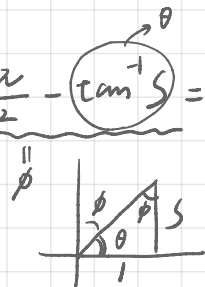
pf: $\int_s^\infty F(u) du = \int_s^\infty \left[\int_0^\infty f(t) e^{-ut} dt \right] du$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-ut} du \right] dt = \int_0^\infty f(t) \frac{e^{-st}}{-t} \Big|_s^\infty dt$$

$$= \int_0^\infty f(t) \frac{0 - e^{-st}}{-t} dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

example $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

$$= \int_s^\infty \frac{1}{u^2 + 1} du = \tan^{-1} u \Big|_s^\infty = \tan^{-1} \infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}$$



unit impulse & Dirac Delta function

$$\delta(t) = \lim_{\epsilon \rightarrow 0} p_{\epsilon}(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u(t) - u(t-\epsilon)) = \begin{cases} 0 & t \neq 0^+ \\ \infty & t = 0^+ \end{cases}$$

性質

$$\begin{cases} 1. \int_0^{\infty} \delta(t) dt = 1 \\ 2. g(t) \text{ 為連續函數} \begin{cases} \int_0^{\infty} g(t) \delta(t) dt = g(0^+) \\ g(t) \delta(t) = g(0^+) \delta(t) \end{cases} \\ 3. u'(t) = \delta(t) \end{cases}$$

$$\mathcal{L}\{\delta(t)\}$$

$$pf: \mathcal{L}\{\delta(t)\} = \mathcal{L}\{u'(t)\} = \mathcal{L}\left\{\lim_{\epsilon \rightarrow 0} \frac{u(t) - u(t-\epsilon)}{\epsilon}\right\} = \lim_{\epsilon \rightarrow 0} \left[\frac{\frac{1}{s} - e^{-\epsilon s} \cdot \frac{1}{s}}{\epsilon}\right] = \lim_{\epsilon \rightarrow 0} \left[\frac{1 - e^{-\epsilon s}}{s \cdot \epsilon}\right]$$

$$\text{example } \mathcal{L}\{\delta(t-a)\}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{-(-s) e^{-\epsilon s}}{s} = 1$$

$$= \mathcal{L}\left\{\lim_{\epsilon \rightarrow 0} p_{\epsilon}(t-a)\right\}$$

$$= \mathcal{L}\left\{\lim_{\epsilon \rightarrow 0} \frac{u(t-a) - u(t-a-\epsilon)}{\epsilon}\right\} = \lim_{\epsilon \rightarrow 0} \left[\frac{\frac{1}{s} e^{-as} - \frac{1}{s} e^{-(a+\epsilon)s}}{\epsilon}\right]$$

$$= \lim_{\epsilon \rightarrow 0} \frac{e^{-as} - e^{-as} \cdot e^{-\epsilon s}}{\epsilon s} = \lim_{\epsilon \rightarrow 0} \frac{e^{-as} (1 - e^{-\epsilon s})}{s \epsilon}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{e^{-as} (s e^{-\epsilon s})}{s} = e^{-as}$$

Convolution Integral 折和積分, 卷積

$$f(t), g(t), t > 0,$$

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = g(t) * f(t) = \int_0^t g(\tau) \cdot f(t-\tau) d\tau$$

$$\mathcal{L}\{f(t)\} = F(s), \mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$$

$$pf: \mathcal{L}\left[\int_0^t g(\tau) f(t-\tau) d\tau\right]$$

$$= \int_0^{\infty} e^{-st} \int_0^t g(\tau) f(t-\tau) d\tau dt$$

$$= \int_0^{\infty} g(\tau) \int_{\tau}^{\infty} e^{-st} f(t-\tau) dt d\tau$$

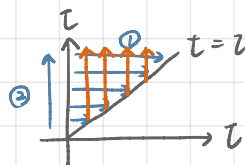
$$\left(\begin{array}{l} \text{令 } x = t - \tau, t = x + \tau, dt = dx \end{array}\right)$$

$$= \int_0^{\infty} g(\tau) \int_0^{\infty} e^{-s(x+\tau)} f(x) dx d\tau$$

$$= \int_0^{\infty} g(\tau) e^{-s\tau} \int_0^{\infty} e^{-sx} f(x) dx d\tau$$

$$= \int_0^{\infty} g(\tau) e^{-s\tau} F(s) d\tau$$

$$= F(s) \cdot G(s) = G(s) \cdot F(s)$$



example $x(t) * h(t)$

$$x(t) = u(t) - u(t-1)$$

$$h(t) = [2u(t) - u(t-1)] + [u(t-1) - u(t-2)]$$

$$= 2u(t) - u(t-1) - u(t-2)$$

$$X(s) = \mathcal{L}\{x(t)\} = \frac{1}{s} - \frac{e^{-s}}{s}$$

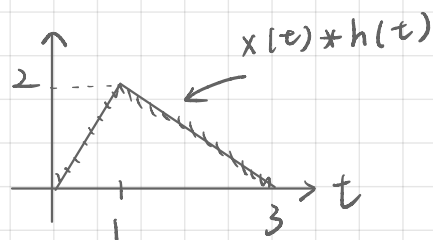
$$H(s) = \mathcal{L}\{h(t)\} = \frac{2}{s} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$X(s) \cdot H(s) = \frac{1}{s^2} (2 - 3e^{-s} + e^{-2s})$$

$$x(t) * h(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2} - \frac{3e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right\}$$

$$= 2t u(t) - 3(t-1) u(t-1) + (t-2) u(t-2)$$

$$\begin{cases} 2t & 0 \leq t < 1 \\ 2t + (-3t + 3) = 3 - t & 1 \leq t < 2 \\ (3 - t) + (t - 2) = 1 & t \geq 2 \end{cases}$$



example using convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = ?$$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} = F(s) \cdot G(s)$$

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t) = \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{G(s)\}$$

$$\frac{1}{s^2} = F(s), \quad \frac{1}{(s+1)^2} = G(s)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t \cdot e^{-t}$$

$$g(t) * f(t) = \int_0^t (t-\tau) e^{-\tau} d\tau = \int_0^t (t\tau e^{-\tau} - \tau^2 e^{-\tau}) d\tau$$

$$= \int_0^t t\tau e^{-\tau} d\tau - \int_0^t \tau^2 e^{-\tau} d\tau$$

$$= t \int_0^t \tau e^{-\tau} d\tau - \left[(-\tau^2 e^{-\tau}) \Big|_0^t + 2 \int_0^t \tau e^{-\tau} d\tau \right]$$

$$= t^2 e^{-t} \Big|_0^t + (t+2) \int_0^t \tau e^{-\tau} d\tau$$

$$= t^2 e^{-t} + (t+2)(-te^{-t} - e^{-t} + 1) = (t+2)e^{-t} + (t-2)$$

部分分式展開

$$f(s) = \frac{P(s)}{Q(s)}$$

$Q(s) = 0$ 之根之情形

1. $P(s), Q(s)$ 係較為 real

2. $P(s), Q(s)$ 無共同因式

3. $Q(s)$ 之次數比 $P(s)$ 之次數高

• 情況 1 $Q(s) = 0$ 具有 n 個不同的一次根 $a_k, k=1, 2, \dots, n$.

$$\frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \quad A=2, B=-1$$

• 情況 2 若 $Q(s) = 0$ 具有 a_k 之 m 重根

$$\frac{2s^2+3}{(s+1)^3} = \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{C_3}{(s+1)^3}$$

$$C_3 = 2s^2+3 \big|_{s=-1}$$

$$C_2 = \frac{d}{ds}(2s^2+3) \big|_{s=-1}$$

$$C_1 = \frac{d^2}{ds^2}(2s^2+3) \big|_{s=-1}$$

• 情況 3 若 $Q(s) = 0$ 含有共軛複數根

$Q(s) = 0$, $Q(s)$ 有 $Xs^2 + Ys + Z$

$$s = \frac{-Y \pm \sqrt{Y^2 - 4XZ}}{2X}$$

$$s = a \pm ib, i = \sqrt{-1}$$

$$(s - (a+ib))(s - (a-ib))$$

$$= (s-a-ib)(s-a+ib)$$

$$= (s-a)^2 - (ib)^2 = (s-a)^2 + b^2$$

$$\frac{P(s)}{Q(s)} = \frac{As+B}{(Xs^2+Ys+Z)} + H(s) = \frac{As+B}{(s-a)^2+b^2} + H(s)$$

$$\frac{P(s)}{Q(s)} [(s-a)^2+b^2] = As+B+H(s)[(s-a)^2+b^2]$$

$$\left. \frac{\frac{P(s)}{Q(s)}}{(s-a)^2+b^2} \right|_{s=a+ib} = A(a+ib) + B + 0$$

$$\text{example} = \frac{s^2+1}{s(s^2+9)} = \frac{As+B}{s^2+9} + \frac{C}{s}$$

$$s^2+9=0, s = \pm 3i$$

$$\frac{(s^2+1)(s^2+9)}{s(s^2+9)} = As+B + \frac{C(s^2+9)}{s}, \quad \frac{s^2+1}{s} = As+B$$

$$s = 3i \Rightarrow \frac{-9+1}{3i} = 3Ai+B$$

$$-8 = -9A + 3B \cdot i$$

$$A = \frac{8}{9}, B = 0$$

$$s=0, \quad \frac{1}{9} = C$$

example

$$(1) \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2s + 2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{(s+1)-1}{(s+1)^2 + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+1}{(s+1)^2 + 1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 1} \right\} = e^{-t} \cos t - e^{-t} \sin t$$

$$(2) \mathcal{L}^{-1} \left\{ \frac{(s-1)e^{-s}}{s(s+1)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \left(\frac{A}{s} + \frac{B}{s+1} \right) e^{-s} \right\} = \mathcal{L}^{-1} \left\{ \left(\frac{1}{s} + \frac{2}{s+1} \right) e^{-s} \right\}$$

$A = -1, B = 2$

$$= (-1 + 2e^{-(t-1)}) u(t-1)$$

example $\mathcal{L}^{-1} \left\{ \frac{(s+1)e^{-2s}}{(s^2+4)(s-1)^2} \right\}$

$$\Rightarrow \frac{s+1}{(s^2+4)(s-1)^2} = \frac{A(s-1)}{(s-1)^2} + \frac{B(s-1)}{(s-1)^2} + \frac{Cs+D}{s^2+4}$$

$$s=1 \quad \frac{2}{5} = 0 + B + 0, \quad B = \frac{2}{5}$$

$$\text{For } A = ? \quad \frac{d}{ds} \left(\frac{s+1}{s^2+4} \right) \Big|_{s=1} = A + \frac{d}{ds} \left(\frac{(Cs+D)(s-1)^2}{s^2+4} \right) \Big|_{s=1}, \quad A = \frac{(s^2+4) \cdot 1 - (s+1) \cdot 2s}{(s^2+4)^2} \Big|_{s=1} = \frac{1}{25}$$

$$\frac{s+1}{(s^2+4)(s-1)^2} = \frac{A(s-1)}{(s-1)^2} + \frac{B(s-1)}{(s-1)^2} + \frac{Cs+D}{s^2+4}$$

$$s^2+4 = \pm 2i \quad \frac{s+1}{(s-1)^2} \Big|_{s=2i} = 0 + 0 + Cs + D \Big|_{s=2i}$$

$$\frac{2i+1}{(2i-1)^2} = C(2i) + D, \quad C = \frac{-1}{25}, \quad D = \frac{-1}{25}$$

$$\mathcal{L}^{-1} \left\{ \left(\frac{1}{s-1} + \frac{\frac{10}{25}}{(s-1)^2} + \frac{-\frac{1}{25}s + \frac{-11}{25}}{s^2+4} \right) e^{-2s} \right\}$$

$$\frac{1}{25} (e^{t-2} + 10e^{t-2} - \cos 2(t-2) - \frac{11}{2} \sin 2(t-2)) \cdot u(t-2)$$

Solution of Initial value problem using the Laplace Transform

導出數之 Laplace Transform

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\text{pf: } \int_0^\infty f'(t) e^{-st} dt = \int_0^{t_1} f'(t) e^{-st} dt + \int_{t_1}^{t_2} f'(t) e^{-st} dt + \dots + \int_{t_n}^\infty f'(t) e^{-st} dt$$

$$u = e^{-st}, \quad dv = f(t) dt$$

$$du = -s e^{-st} dt, \quad v = f(t)$$

$$\Rightarrow f(t) e^{-st} \Big|_0^{t_1} + s \int_0^{t_1} f(t) e^{-st} dt$$

$$= f(t) e^{-st} \Big|_0^{t_1} + f(t) e^{-st} \Big|_{t_1}^{t_2} + \dots + f(t) e^{-st} \Big|_{t_n}^\infty + s \int_{t_0}^{t_1} f(t) e^{-st} dt + \dots + s \int_{t_n}^\infty f(t) e^{-st} dt$$

$$= f(t) e^{-st} \Big|_0^\infty + s \int_0^\infty f(t) e^{-st} dt = 0 - f(0) + sF(s)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

pf. 取 $n=2$

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= \mathcal{L}\{g'(t)\} = s \cdot \mathcal{L}\{g(t) - g(0)\} \\ &= s \cdot \mathcal{L}\{f'(t)\} - g(0) \\ &= s [s F(s) - f(0)] - f'(0) \end{aligned}$$

$$\begin{aligned} \text{令 } g(t) &= f'(t) \\ g'(t) &= f''(t) \end{aligned}$$

example $\frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 6y = 6 \sin 2t, y(0) = 0, \frac{dy}{dt} = 0$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s Y(s) - y(0) = s Y(s)$$

$$\mathcal{L}\left\{\frac{d^2 y}{dt^2}\right\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s)$$

$$s^2 Y(s) + 7s Y(s) + 6Y(s) = \frac{12}{s^2 + 4}$$

$$Y(s) = \frac{12}{(s^2 + 4)(s^2 + 7s + 6)} = \frac{12}{(s^2 + 4)(s+6)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+6} + \frac{D}{s+1}$$

$$\begin{cases} A = \frac{-21}{50} \\ B = \frac{23}{50} \\ C = \frac{-3}{50} \\ D = \frac{12}{25} \end{cases}$$

$$y(t) \Rightarrow \frac{-21}{50} \cos 2t + \frac{3}{50} \sin 2t - \frac{3}{50} e^{-6t} + \frac{12}{25} e^{-t}$$

Laplace 轉換之性質

Diff eqs with polynomial coefficients

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{t \cdot f(t)\} = (-1) \frac{d}{ds} F(s)$$

pf: $F(s) = \int_0^\infty f(t) e^{-st} dt$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty \frac{d}{ds} (f(t) e^{-st}) dt = \int_0^\infty -t f(t) e^{-st} dt = -\mathcal{L}\{t \cdot f(t)\}$$

$$\mathcal{L}\{f(t)\} = F(s)$$

$$F(s) \text{ 可微 } n \text{ 次 } \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$n=2$ 來證

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds} \left(\frac{d}{ds} F(s) \right)$$

$$\Rightarrow \text{令 } \frac{d}{ds} F(s) = G(s) \quad \mathcal{L}\left\{\frac{d}{ds} F(s)\right\} = \mathcal{L}\{G(s)\} \quad -t \cdot f(t) = g(t)$$

$$\frac{d}{ds} G(s) = -\mathcal{L}\{t \cdot g(t)\} = -\mathcal{L}\{t(-t \cdot f(t))\} = (-1)^2 \mathcal{L}\{t^2 f(t)\}$$

example

$$(I) \mathcal{L}\{t \cdot \underbrace{f(t)}_{\text{mat}}\} = -\frac{d}{ds} F(s)$$

$$= -\frac{d}{ds} \left(\frac{a}{s^2 + a^2} \right) = -\frac{d}{ds} (a \cdot (s^2 + a^2)^{-1})$$

$$= -a(-1)(s^2 + a^2)^{-2} \cdot 2s = \frac{2as}{(s^2 + a^2)^2}$$

$$(II) \mathcal{L}\{t \cdot e^{3t}\} = -\frac{d}{ds} \frac{1}{s-3} = \frac{1}{(s-3)^2}$$

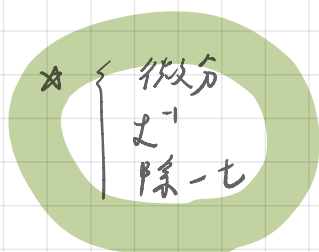
$$(III) \mathcal{L}^{-1}\left\{ \underbrace{\ln\left(\frac{s+a}{s-a}\right)}_{F(s)} \right\}$$

$$F(s) = \ln\left(\frac{s+a}{s-a}\right) = \ln(s+a) - \ln(s-a)$$

$$\textcircled{1} \frac{d}{ds} F(s) = \frac{1}{s+a} - \frac{1}{s-a}$$

$$\textcircled{2} \mathcal{L}^{-1}\left\{ \frac{1}{s+a} - \frac{1}{s-a} \right\} = e^{-at} - e^{at}$$

$$\textcircled{3} f(t) = \frac{e^{-at} - e^{at}}{-t}$$



example $y'' - y' - y = 0, y(0) = 0, y'(0) = 3$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{t \cdot \underbrace{y'}_{f(t)}\} = -\frac{d}{ds} \underbrace{(sY(s) - y(0))}_{F(s)} = -\frac{d}{ds}(sY(s))$$

$$\mathcal{L}\{t \cdot y''\} = -\frac{d}{ds}(s^2 Y(s) - s y'(0) - y'(0)) = -\frac{d}{ds}(s^2 Y(s) - 3)$$

对上述作 Laplace Transform

$$-\frac{d}{ds}[s^2 Y(s) - 3] - (-\frac{d}{ds}[sY(s)]) - Y(s) = 0$$

$$-(2sY(s) + s^2 Y'(s)) + Y(s) + sY'(s) - Y(s) = 0$$

$$(-s^2 + s)Y'(s) - 2sY(s) = 0$$

$$\frac{dY(s)}{ds} + \frac{2s}{s(s-1)} Y(s) = 0 \quad \mu = e^{\int \frac{2}{s-1} ds} = (s-1)^2$$

$$\frac{d}{ds}(Y(s) \cdot (s-1)^2) = 0, Y(s)(s-1)^2 = C, Y(s) = \frac{C}{(s-1)^2}$$

取逆转换 $y(t) = \mathcal{L}^{-1}\{Y(s)\} = C t \cdot e^t$

$$y(0) = 0, y'(0) = 3 = C e^0 + C t e^t = C = 3$$

$$\Rightarrow y(t) = 3te^t$$

Laplace Transform Solution system

example $x'' - 2x' + 3y' + 2y = 4$ $x(0) = x'(0) = y(0) = 0$
 $xy' - x' + 3y = 0$

$$\begin{cases} s^2 x(s) - 2s x(s) + 3Y(s) + 2Y(s) = \frac{4}{s} \\ 2s Y(s) - s x(s) + 3Y(s) = 0 \end{cases} \quad \begin{bmatrix} (s^2 - 2s) & (3s + 2) \\ (-s) & (2s + 3) \end{bmatrix} \begin{bmatrix} x \\ Y \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 0 \end{bmatrix}$$

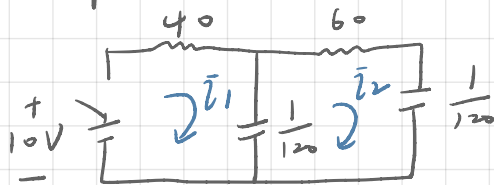
$$X = \frac{\frac{4}{s}(2s+3)}{(s^2-2s)(2s+3)+s(3s+2)} = \frac{4s+6}{s^2(s+2)(s-1)} = \frac{\frac{-7}{2}}{s} + \frac{-3}{s^2} + \frac{\frac{1}{6}}{s+2} + \frac{\frac{10}{3}}{s-1}$$

$$Y = \frac{s \cdot \frac{4}{s}}{(s^2-2s)(2s+3)+s(3s+2)} = \frac{2}{s(s+2)(s-1)} = \frac{\frac{A}{s}}{-1} + \frac{\frac{B}{s+2}}{\frac{1}{3}} + \frac{\frac{C}{s-1}}{\frac{2}{3}}$$

$$x(t) = \frac{-7}{2} - 3t + \frac{1}{6}e^{-2t} + \frac{10}{3}e^t$$

$$y(t) = -1 + \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$$

example $\bar{i}_1, \bar{i}_2$



$$\bar{i} = \frac{dq}{dt}$$

$$q_1(0) = 0, q_2(0) = 0$$

$$40\bar{i}_1 + \frac{q_1 - q_2}{\frac{1}{120}} = 10$$

$$\begin{cases} 40\bar{i}_1 + (120)(q_1 - q_2) = 10 \\ 60\bar{i}_2 + (120)(q_2 - q_1) + 120q_2 = 0 \end{cases}$$

$$\begin{cases} 40 \frac{dq_1}{dt} + 120(q_1 - q_2) = 10 \\ 60 \frac{dq_2}{dt} + 120(q_2 - q_1) + 120q_2 = 0 \end{cases} \quad \begin{cases} 40sQ_1 + 120(Q_1 - Q_2) = \frac{10}{s} \\ 60sQ_2 + 120Q_2 + 120(Q_2 - Q_1) = 0 \end{cases}$$

$$\begin{cases} (s+3)Q_1 - 3Q_2 = \frac{1}{4s} \\ 2Q_1 + (-s-4)Q_2 = 0 \end{cases} \quad \begin{bmatrix} s+3 & -3 \\ 2 & -s-4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4s} \\ 0 \end{bmatrix}$$

$$Q_1 = \frac{\frac{1}{6}}{s} + \frac{-\frac{1}{60}}{s+6} + \frac{-\frac{3}{20}}{s+1}$$

$$Q_2 = \frac{\frac{1}{12}}{s} + \frac{\frac{1}{60}}{s+6} + \frac{-\frac{1}{10}}{s+1}$$

$$\Rightarrow \begin{aligned} q_1 &= \frac{1}{6} - \frac{1}{60}e^{-6t} - \frac{3}{20}e^{-t} \\ q_2 &= \frac{1}{12} + \frac{1}{60}e^{-6t} - \frac{1}{10}e^{-t} \end{aligned}$$

$$\bar{i}_1 = \frac{dq_1}{dt} = \frac{1}{10}e^{-6t} + \frac{3}{20}e^{-t}$$

$$\bar{i}_2 = \frac{dq_2}{dt} = \frac{-1}{10}e^{-6t} + \frac{1}{10}e^{-t}$$

power series 幂级数

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 (x-x_0)^0 + a_1 (x-x_0)^1 + a_2 (x-x_0)^2 + \dots + a_{\infty} (x-x_0)^{\infty}$$

converges for $x = x_0$

example:

$$\sum_{n=0}^{\infty} n! x^n = \sum_{n=0}^{\infty} n! (x-0)^n$$

converges for $x = 0$
diverges for $x \neq 0$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$x = x_0$ 收敛
 $x = x_1$ 亦收敛 $x_1 \neq x_0$
 $\rightarrow x_0 = 3, x_1 = 4$

$\rightarrow |x-x_0| < |x_1-x_0| \Rightarrow$ 都收敛

$$|x-3| < |4-3| = 1$$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ for convergence}$$

1. converge only for $x = x_0$ ($r=0$)
2. converge for all real numbers x ($r=\infty$)
3. positive number r radius of convergence 收敛半径

if $|x-x_0| < r$ 收敛
if $|x-x_0| > r$ 发散

$$x_0 - r < x < x_0 + r$$

收敛区间 open interval of convergence
 $(x_0 - r, x_0 + r)$

ratio test $\sum_{n=0}^{\infty} b_n \quad b_n \neq 0$

$$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = L \begin{cases} L < 1 \rightarrow \text{converges} \\ L > 1 \rightarrow \text{diverges} \\ L = 1 \rightarrow \text{no conclusion} \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \text{ ratio test: } \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{(n+1)}}{n! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1) \cdot x}{1} \right| = \begin{cases} \text{if } x \neq 0, \infty > 1 \\ \text{if } x = 0, 0 < 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ ratio test } \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n! x^{n+1}}{(n+1)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

$\rightarrow$ converges
 $r = \infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)9^n} (x-2)^{2n}$$

ratio test $\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (x-2)^{2(n+1)}}{(n+2)9^{n+1}}}{\frac{(-1)^n (x-2)^{2n}}{(n+1)9^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{(n+2) \cdot 9} (x-2)^2 \right| = (x-2)^2 \lim_{n \rightarrow \infty} \left| \frac{n+1}{9(n+2)} \right| = (x-2)^2 \cdot \frac{1}{9}$

$$(x-2)^2 < 9, |x-2| < 3, 3 < x-2 < -3, -1 < x < 5$$

收敛半径 = 3, 收敛区间 $(-1, 5)$

Analytic function

可微多次

f is analytic at x_0 , if f has a power series represent about x_0

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$y' + p(x)y = q(x), y(x_0) = y_0$$

p, q 可解析 $\Rightarrow$ 解亦是在 x_0 可解析 $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, a_n = \frac{1}{n!} y^{(n)}(x_0)$

example

$$y' + e^x y = x^2, y(0) = 4$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0) x^n = y(0) + \frac{1}{1!} y'(0) x + \frac{1}{2!} y''(0) x^2 + \frac{1}{3!} y'''(0) x^3 + \dots$$

$$y'(0) + e^0 y(0) = 0^2 = -y(0) = -4$$

$$\text{对 } x \text{ 求导得 } y'' + (e^x y' + e^x y) = 2x$$

$$y''(0) = -y'(0) - y(0) = -(-4) - 4 = 0$$

$$y''' + e^x y'' + e^x y' + e^x y = 2, y'''(0) = 6$$

$$y(x) = 4 - 4x + 0 + \frac{1}{3!} \cdot 6 \cdot x^3 + \frac{1}{4!} 2x^4 = 4 - 4x + x^3 + \frac{1}{12} x^4 + \dots$$

let p, q and f be analytic at x_0 , the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$

$y(x_0) = A, y'(x_0) = B$, has a unique sol that is also analytic at x_0

$$\text{example: } y'' - xy' + e^x y = 4, y(0) = 1, y'(0) = 4$$

since $-x, e^x, 4$ are analytic at 0

series sol expanded about 0

$$y(x) = y(0) + y'(0)x + \frac{1}{2!} y''(0)x^2 + \frac{1}{3!} y'''(0)x^3 + \dots$$

$$y''(0) = ?$$

$$y''(0) = 0 \cdot y'(0) - e^0 y(0) + 4 = 3$$

$$\text{对 } x \text{ 求导得 } y''' - (xy'' + y') + (e^x y' + e^x y) = 0$$

$$y'''(0) = 0 \cdot y''(0) + y'(0) - e^0 y'(0) - e^0 y(0) = -1$$

$$y(x) = 1 + 4x + \frac{3}{2}x^2 - \frac{1}{6}x^3 + \dots$$

Power series sol using "Recurrence relation"

example: $y'' + x^2 y = 0$, expanded about 0

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = 2a_2 + 6a_3 x + \dots + n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$\text{代入 ODE } y'' + x^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$\xrightarrow{\frac{1}{2} m = n-2, n = m+2}$

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{m=2}^{\infty} a_{m-2} x^m$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[(0+2)(0+1) \cdot a_2 x^0 + 3 \cdot 2 \cdot a_3 x]$$

$$\sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} + a_{n-2}] x^n = 0$$

$$2a_2 + 6a_3 x + \text{ } = 0$$

$$a_2 = 0, a_3 = 0$$

$$(n+2)(n+1) a_{n+2} + a_{n-2} = 0, n=2, 3, 4, \dots$$

$$a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)} \text{ for } n=2, 3, \dots$$

$$n=2, a_4 = -\frac{a_0}{4 \cdot 3} = -\frac{1}{12} a_0$$

$$n=3, a_5 = -\frac{a_1}{5 \cdot 4} = -\frac{1}{20} a_1$$

$$n=4, a_6 = 0$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x + 0 + 0 - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + 0 + \dots$$

$$= a_0 (1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 + \dots) + a_1 (x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 + \dots)$$

$$= a_0 y_1 + a_1 y_2 \quad \begin{matrix} a_0 = y(0) \\ a_1 = y'(0) \end{matrix}$$

example $y'' + x'y' + 4y = 1 - x^2$ expanded about 0

$$y(x) = \sum_{n=0}^{\infty} a_n (x-0)^n$$

$$= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} 4 a_n x^n = 1 - x^2$$

$$\left\{ \begin{array}{l} \frac{1}{2} n-2 = m, n = m+2 \\ n+1 = m, n = m-1 \end{array} \right.$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=2}^{\infty} (m-1) a_{m-1} x^m + \sum_{n=0}^{\infty} 4 a_n x^n = 1 - x^2$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 1 - x^2$$

$$\left(\underbrace{2 \cdot 1 \cdot a_2 x^0}_{n=0} + \underbrace{3 \cdot 2 \cdot a_3 x^1}_{n=1} \right) + \left(\underbrace{4 a_0 x^0}_{n=0} + \underbrace{4 a_1 x^1}_{n=1} \right) + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} + (n-1) a_{n-1} + 4 a_n x^n = 1 - x^2$$

$$n=2 \left\{ \begin{array}{l} 2 a_2 + 4 a_0 = 1 \\ 6 a_3 + 4 a_1 = 0 \\ 4 \cdot 3 a_4 + 1 \cdot a_1 + 4 a_2 = -1 \end{array} \right.$$

$$(n+2)(n+1) a_{n+2} + (n-1) a_{n-1} + 4 a_n = 0, n=3, 4, \dots$$

$$a_0 = y(0), a_1 = y'(0)$$

$$a_2 = \frac{1-4a_0}{2} = \frac{1}{2} - 2a_0 \quad a_3 = \frac{-4a_1}{6} = -\frac{2}{3} a_1$$

$$a_4 = \frac{1}{12} (-1 - a_1 - 4a_2) = \frac{1}{12} \left[-1 - a_1 - 4 \left(\frac{1}{2} - 2a_0 \right) \right] = -\frac{1}{4} + \frac{2}{3} a_0 - \frac{1}{12} a_1$$

$$a_{n+2} = \frac{4a_n + (n-1)a_{n-1}}{(n+2)(n+1)} \quad n=3, 4, \dots$$

$$y(x) = a_0 + a_1 x + \left(\frac{1}{2} - 2a_0 \right) x^2 + \left(-\frac{2}{3} a_1 \right) x^3 + \left(-\frac{1}{4} + \frac{2}{3} a_0 - \frac{1}{12} a_1 \right) x^4 + \dots$$

$$= a_0 y_1 + a_1 y_2 + \underline{y_p}$$

Singular point and the method of Frobenius

Ordinary point and singular point

x_0 is an ordinary point

$$p(x)y'' + Q(x)y' + R(x)y = F(x), \quad p(x) \neq 0$$

$$\frac{Q(x)}{p(x)}, \frac{R(x)}{p(x)}, \frac{F(x)}{p(x)} \text{ at } x_0 \text{ are analytic}$$

x_0 is a singular point $\rightarrow$ if x_0 is not an ordinary point

example

$$x^3(x-2)^2 y'' + 5(x+2)(x-2)y' + 3x^2 y = 0$$

$$x=0, \quad x^3(x-2)^2|_{x=0} = 0$$

$$x=2, \quad x^3(x-2)^2|_{x=2} = 0 \quad \rightarrow x=0, 2 \rightarrow \text{singular point}$$

Regular and Irregular singular point

x_0 is regular singular point

$$p(x)y'' + Q(x)y' + R(x)y = 0$$

x_0 is a singular point

$$(x-x_0) \frac{Q(x)}{p(x)}$$

$$(x-x_0)^2 \frac{R(x)}{p(x)} \left. \begin{array}{l} \\ \end{array} \right\} \text{are analytic at } x_0 \rightarrow x_0 \text{ is "regular" singular point}$$

example

$$x^3(x-2)^2 y'' + 5(x+2)(x-2)y' + 3x^2 y = 0$$

singular point $\rightarrow 0, 2$

$$x_0 = 0$$

$$(x-0) \frac{5(x+2)(x-2)}{x^3(x-2)^2} = \frac{5(x+2)}{x^2(x-2)} \Big|_{x=0} \text{ not analytic} \Rightarrow 0 \text{ is irregular singular point}$$

$$x_0 = 2$$

$$(x-2) \frac{5(x+2)(x-2)}{x^3(x-2)^2} \Big|_{x=2} \Rightarrow \text{analytic}$$

$$(x-2)^2 \frac{3x^2}{x^3(x-2)^2} \Big|_{x=2} \Rightarrow \text{analytic}$$

$\Rightarrow 2$ is regular singular point

For regular singular point at x_0 Frobenius series

$$y(x) = \sum_{n=0}^{\infty} C_n (x-x_0)^{n+r} = C_0 (x-x_0)^r + C_1 (x-x_0)^{1+r} + C_2 (x-x_0)^{2+r} + \dots$$

$$y' = C_0 r (x-x_0)^{r-1} + \dots = \sum_{n=0}^{\infty} (n+r) C_n (x-x_0)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n (x-x_0)^{n+r-2}$$

example

$$x^2 y'' + x(\frac{1}{2} + x) y' + (x - \frac{1}{2}) y = 0$$

$$(x-0) \cdot \frac{x(\frac{1}{2} + x)}{x^2} \Big|_{x \rightarrow 0} \text{ 可解析}$$

$$(x-0)^2 \frac{(x - \frac{1}{2})}{x^2} \Big|_{x \rightarrow 0} \text{ 可解析}$$

$\Rightarrow x=0$ is regular singular point

By Frobenius series

$$y(x) = \sum_{n=0}^{\infty} C_n (x-x_0)^{n+r} = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r} + \sum_{n=0}^{\infty} \frac{1}{2} (n+r) C_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r) C_n x^{n+r+1} + \sum_{n=0}^{\infty} C_n x^{n+r+1} - \sum_{n=0}^{\infty} \frac{1}{2} C_n x^{n+r} = 0$$

$$\frac{1}{2} n+1 = m, n = m-1$$

$$\sum_{m=1}^{\infty} 2(m-1+r) C_{m-1} x^{m+r} + \sum_{m=1}^{\infty} C_{m-1} x^{m+r}$$

$$\sum_{n=1}^{\infty} 2(n-1+r) C_{n-1} x^{n+r} + \sum_{n=1}^{\infty} C_{n-1} x^{n+r}$$

$n=0$ case

$$[r(r-1) C_0 + \frac{1}{2} r C_0 - \frac{1}{2} C_0] x^r$$

$$+ \sum_{n=1}^{\infty} [(n+r)(n+r-1) C_n + \frac{1}{2} (n+r) C_n + 2(n+r-1) C_{n-1} + C_{n-1} - \frac{1}{2} C_n] x^{n+r} = 0$$

$$r(r-1) C_0 + \frac{1}{2} r C_0 - \frac{1}{2} C_0 = 0$$

$$C_0 (r(r-1) + \frac{1}{2} r - \frac{1}{2}) = 0$$

$$C_n = - \frac{1 + 2(n+r-1)}{(n+r)(n+r-1) + \frac{1}{2} (n+r) - \frac{1}{2}} C_{n-1}, n=1, 2, 3, \dots$$

assume $C_0 \neq 0$, $r^2 - r + \frac{1}{2} r - \frac{1}{2} = 0$, $r^2 - \frac{1}{2} r - \frac{1}{2} = 0 \rightarrow$ indicial equation

$$(r-1)(r+\frac{1}{2}) = 0$$

for $r=r_1=1$

$$C_n = - \frac{1+2n}{(n+1)n + \frac{1}{2}n + \frac{1}{2} - \frac{1}{2}} C_{n-1} = - \frac{1+2n}{n^2 + \frac{3}{2}n} C_{n-1}, n=1, 2, 3, \dots$$

$$C_1 = - \frac{3}{\frac{5}{2}} C_0 = - \frac{6}{5} C_0$$

$$C_2 = - \frac{5}{\frac{7}{2}} C_1 = - \frac{5}{\frac{7}{2}} \cdot - \frac{6}{5} C_0 = \frac{6}{7} C_0$$

$$C_3 = - \frac{4}{\frac{9}{2}} C_0$$

$$y_1(x) = \sum_{n=0}^{\infty} C_n (x)^{n+1} = C_0 x^1 + C_1 x^2 + C_2 x^3 + \dots$$

$$= C_0 x + (-\frac{6}{5} C_0) x^2 + \frac{6}{7} C_0 x^3 + (-\frac{4}{9} C_0) x^4 + \dots$$

$$\text{For } r=r_2=-\frac{1}{2} \quad C_n \Rightarrow C_n^*$$

$$C_n^* = -\frac{1+2(n-\frac{3}{2})}{(n-\frac{1}{2})(n-\frac{3}{2})+\frac{1}{2}(n-\frac{1}{2})-\frac{1}{2}} C_{n-1}^*, \quad n=1, 2, 3, \dots$$

$$= \frac{2n-2}{n(n-\frac{3}{2})} C_{n-1}^*, \quad n=1, 2, 3, \dots$$

$$C_1^* = 0$$

$$C_2^* = 0$$

$$\vdots$$

$$C_n^* = 0 \quad \text{for } n=1, 2, 3, \dots$$

$$y_2(x) = \sum_{n=0}^{\infty} C_n^* X^{n-\frac{1}{2}} = C_0^* X^{\frac{1}{2}}$$

$$y = C_1 y_1 + C_2 y_2 = C_0 y_1 + C_0^* y_2$$

Maclaurin Series	I
$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} X^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} X^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$[-1, 1]$
$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} X^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$	$(-\infty, \infty)$
$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} X^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$	$(-\infty, \infty)$
$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$(-1, 1]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} X^n = 1 + x + x^2 + x^3 + \dots$	$(-1, 1)$