Supplementary material:
Deriving the uncertainty inequality

Yi-Wen Liu

20 Oct 2015

Suppose that $x(t) \leftrightarrow X(\omega)$ is a Fourier transform pair. For simplicity, let us assume that $x(t)$ is infinitely smooth (i.e., $x^{(n)}(t)\ \exists, \forall n$) and $\int |x(t)|^2 dt < \infty$. Without loss of generality, let us assume that $\int |x(t)|^2 dt = 1$. From Parseval's theorem, we have $\int |X(\omega)|^2 d\omega = 2\pi$.

Now, the uncertainty principle is about treating the power $|x(t)|^2$ and power density spectrum $|X(\omega)|^2$ as if they are probability density functions, so we can talk about their *moments*. Define the first moment in the time and frequency domains as follow,

$$\bar{t} = \int t|x(t)|^2 dt,$$

and

$$\bar{\omega} = \frac{\int \omega |X(\omega)|^2 dt}{\int |X(\omega)|^2 dt} = \frac{1}{2\pi} \int \omega |X(\omega)|^2 dt.$$

The "variances" are defined accordingly,

$$(\Delta t)^2 = \int (t - \bar{t})^2 |x(t)|^2 dt,$$

and

$$(\Delta \omega)^2 = \frac{1}{2\pi} \int (\omega - \bar{\omega})^2 |X(\omega)|^2 dt.$$

Below is the uncertainty principle in the form of time-frequency duality.

THEOREM (Uncertainty):

For any $x(t) \leftrightarrow X(\omega)$ defined above, we have $(\Delta t)(\Delta \omega) \geq \frac{1}{2}$.

Proof: Define $\tilde{x}(t) = x(t - \bar{t})e^{-j\bar{\omega}t}$, and the first moments of $\tilde{x}(t)$ becomes 0 in both the time and the frequency domain, but the second moments of $\tilde{x}(t)$ remain the same as those of $x(t)$. So, it suffices to consider $(\Delta t)^2 = \int t^2 |\tilde{x}(t)|^2 dt$ and $(\Delta \omega)^2 = \frac{1}{2\pi} \int \omega^2 |\tilde{X}(\omega)|^2 d\omega$.

Then,

$$\begin{aligned}
(\Delta t)^2(\Delta\omega)^2 &= \Big(\int t^2|\tilde{x}(t)|^2 dt\Big)\Big(\frac{1}{2\pi}\int \Big|j\omega\tilde{X}(\omega)\Big|^2 d\omega\Big) \\
&= \Big(\int t^2|\tilde{x}(t)|^2 dt\Big)\Big(\int \Big|\frac{d\tilde{x}}{dt}\Big|^2 dt\Big) \\
&\geq \Big(\int t\,|\tilde{x}(t)|\cdot\Big|\frac{d\tilde{x}}{dt}\Big|\,dt\Big)^2 \\
&= \Big(\int t\cdot\frac{1}{2}d\,|\tilde{x}(t)|^2\Big)^2 \\
&= \frac{1}{4}\Big(t|\tilde{x}(t)|^2\Big|_{-\infty}^{\infty} - \int |\tilde{x}(t)|^2 dt\Big)^2 \\
&= \frac{1}{4}.
\end{aligned}$$

Hence we have $(\Delta t)(\Delta\omega) \geq \frac{1}{2}$. Note that $\lim_{t\to\pm\infty} t|\tilde{x}(t)|^2 = 0$ has to hold for $\int_{-\infty}^{\infty}|\tilde{x}(t)|^2 dt$ to be integrable.

Note: In certain literature, the frequency $f = \frac{\omega}{2\pi}$ is used, so $(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$.