CS 2336
Discrete Mathematics

Lecture 6

Proofs: Pigeonhole Principle

Outline

- What is a Pigeonhole Principle ?
- The Generalized Version

Pigeonhole Principle

- Suppose that there are 10 pigeons, and there are 9 pigeonholes
- In the evening, each pigeon will pick one of the pigeonholes to stay
- What will happen?

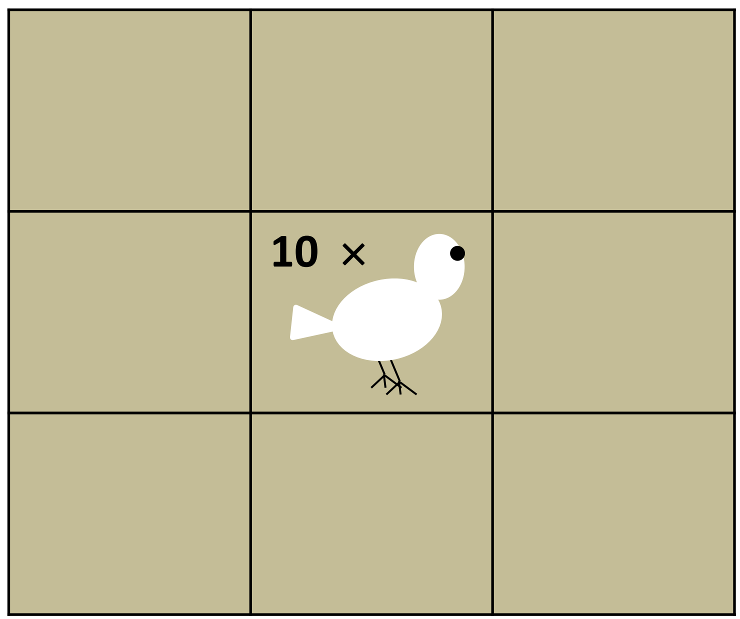
Pigeonhole Principle

- One possible scenario is as follow :

- Here, one of the pigeonholes has 2 pigeons

Pigeonhole Principle

- We may also find ALL the pigeons stay in one pigeonhole (though it is unlikely to happen)



Pigeonhole Principle

- There are many other situations (how many?)
- But in any situation, we can conclude the following :

In one of the pigeonholes, there will be at least two pigeons

- How to show that ?

Pigeonhole Principle

- More generally, we have the following :

> Pigeonhole Principle :
>
> If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one of the boxes will contain two ore more objects

Pigeonhole Principle

- Proof :

  Suppose on the contrary that the proposition is false. Then, we have the case that

  (i) $k + 1$ objects are placed into k boxes, and

  (ii) no boxes contain two or more objects.

  From (ii), it follows that the total number of objects is at most k (since each box has 0 or 1 objects). Thus, a contradiction occurs (where?).

Examples

- Ex 1 :

  Show that there is a number of the form 11…1 that is a multiple of 2013.

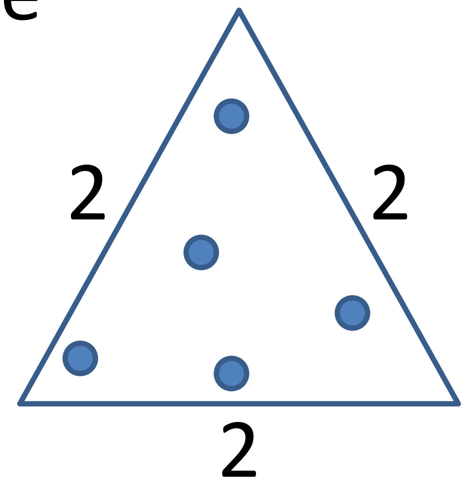
- Ex 2 :

  Show that in any 51 integers chosen from 1 to 100, we can find two of them, such that one divides the other.

Examples

- Ex 3 :

  Suppose that in a given equilateral triangle, with length = 2 units on each side, five points are placed inside



Show that two of the points are ≤ 1 unit apart

Examples

- Ex 4 (A very difficult example):
  Show that in a sequence of $n^2 + 1$ distinct numbers, we can find an increasing or a decreasing subsequence of length $n + 1$

  For instance, say $n = 2$

  (i) 1, 3, 5, 2, 4 ➔ subsequence 1, 2, 4

  (ii) 4, 5, 1, 3, 2 ➔ subsequence 5, 3, 2

  (iii) 2, 4, 1, 5, 3 ➔ subsequence 2, 4, 5

Examples

- Ex 5 (Another difficult one):

Ron recalled :

"It was April in my first year at Hogwarts.
I have been practicing hard with the spell *Wingardium Leviosa*, casting it at least once per day, and 45 times in the month."

Show that in some consecutive days in that month, Ron has casted the spell exactly 14 times.

Generalized Pigeonhole Principle

- In fact, we can generalize the Pigeonhole Principle further :

> Generalized Pigeonhole Principle :
>
> If k is a positive integer and N objects are placed into k boxes, then at least one of the boxes will contain $\lceil N/k \rceil$ or more objects.

Here, $\lceil x \rceil$ is called the ceiling function, which represents the round-up value of x

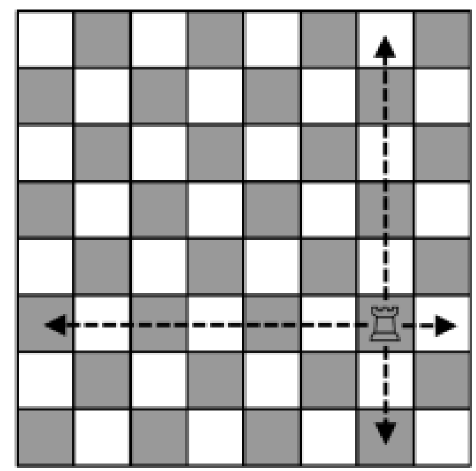
Examples

- Ex 1 :

  Show that among all 80+ students in our class, 7 or more are born in the same month.

- Ex 2 :

  Show that if 33 rooks are placed in a regular 8×8 chessboard, at least 5 of them cannot attack each other.



Examples

- Ex 2 (Solution) :

1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7
7	8	1	2	3	4	5	6
6	7	8	1	2	3	4	5
5	6	7	8	1	2	3	4
4	5	6	7	8	1	2	3
3	4	5	6	7	8	1	2
2	3	4	5	6	7	8	1

We can find 5 or more rooks on cells with the same label, and thus not attacking each other

Examples

- Ex 3 (A difficult one) :

  Show that among six people, where each pair are either friends or enemies, there exist either 3 mutual friends or 3 mutual enemies (or both).