# CS2336 Discrete Mathematics

Exam 1
October 28, 2019 (10:10–12:30)

**Answer all 7 questions. Total marks = 105. Maximum score = 100. For all the proofs, if it is incomplete, large portion of marks may be deducted.**

1. Consider the proposition $(p \wedge \neg q) \leftrightarrow (p \vee q)$.

   (15%) Find an equivalent proposition that is as short as possible. Show your steps.

2. Given a collection of premises, say $p, q, r, s, \ldots$, we say a premise $p$ is *redundant* if it is the case that when all premises except $p$ are true would imply $p$ is true. So, $p$ is redundant if

$$(q \wedge r \wedge s \wedge \cdots) \rightarrow p$$

   is always true.

   Intuitively, a redundant premise will not give us extra knowledge, so we hope to remove it. If after removal, the remaining premises still contain a redundant premise, we will continue to remove it. Our goal is to remove as many redundant premises (so as to keep as few premises in the end) as possible.

   **Example:** Consider the premises (i) $p \leftrightarrow q$, (ii) $p \rightarrow q$, (iii) $q \rightarrow p$. It is easy to check that $p \leftrightarrow q$ is redundant, since if the remaining two premises are true, then $p \leftrightarrow q$ is true. So, we can just keep premises (ii) and (iii).

   However, we can also say both (ii) and (iii) are redundant, because if we know $p \leftrightarrow q$ is true, then both the remaining premises are true. So, we can just keep premise (i), and remove the other two.

   In this example, we just need to keep one premise in the best case.

   (a) (10%) Consider the premises (i) $p$, (ii) $q$, (iii) $p \leftrightarrow q$, (iv) $p \rightarrow q$. What is the minimum number of premises that we need to keep, and which one(s)? Justify your answer.

   (b) (10%) Consider the premises (i) $p \rightarrow q$, (ii) $r \rightarrow (p \rightarrow q)$, (iii) $(p \rightarrow r) \vee ((\neg r) \rightarrow q)$. What is the minimum number of premises that we need to keep, and which one(s)? Justify your answer.

3. (15%) Prove or disprove:

   Let $a$ and $b$ be integers. If $a + b$ is a multiple of 3, then $a^3 + b^3$ is a multiple of 3.

4. (15%) Find three different pairs of integers $n$ and $m$ such that $2^n = 3 + 5^m$.

5. Consider the equation $z^{13} - z^2 - 15015 = 0$.

   (a) (10%) Show that the equation does not have any integral root.

   *Hint:* Show that for any $z$ that satisfies the above equation, (i) $z$ cannot be an odd number, and (ii) $z$ cannot be an even number.

   (b) (10%) Show that the equation does not have any rational root.

6. (Adapted from a logical puzzle in an online competition)

   Raymond is visiting the famous country, *Pureland*, where citizens there are either honest (always tell the truth) or dishonest (always lie). Raymond sees three people, let us identify them as $A$, $B$, and $C$, and chats with them. Suddenly, one of them said "$A$ and $B$ are liars", and then another one of them said "$A$ and $C$ are liars."

   (15%) How many liars are there among these three people? Justify your answer.

7. (Extremely Challenging: Estimated time to solve is more than 1 hour)

   Two super smart boys, Sam and Peter, are present in a room. Teacher Hans goes to them, one by one, secretly telling each of them something. Now, Teacher Hans says: I have chosen two distinct integers, $x$ and $y$, such that $1 < x < y$ and $x + y \leq 65$. I have just told Sam the sum $x + y$, and told Peter the product $xy$.

   Next, the boys begin the following interesting conversation:

   Peter: I don't know the numbers $x$ and $y$.
   Sam: I already knew that you didn't know.
   Peter: Oh, now I know $x$ and $y$.
   Sam: Oh, I also know $x$ and $y$ now.

   (5%) What are the numbers $x$ and $y$? Justify your answer.