CS2336 Discrete Mathematics

Exam 1
October 22, 2018 (10:10–12:30)

Answer all questions. Total marks = 110. Maximum score = 100. For all the proofs, if it is incomplete, large portion of marks may be deducted.

1. (20%) Consider the following two propositions:

   - $p \to r$
   - $(p \wedge q) \to r$

   Peter says that one of the propositions can be a premise, while the other is a conclusion of that premise.

   Is Peter correct? Explain your answer clearly.

2. (15%) Use logical equivalences and rules of inferences to show that the following arguments are valid. Refer to the last page for some common equivalences and rules.

   - Premises: $\forall x P(x)$, $\neg \forall x Q(x)$,
   - Conclusion: $\neg \forall x (P(x) \to Q(x))$

3. (15%) Prove or disprove:

   If x is the difference of two distinct square numbers and $x \geq 2$, then x is not a prime.

4. (15%) Show that there exist positive integers n and m such that n^2 divides m^3, but n does not divide m.

5. (20%) In the 17th century, Pierre de Fermat claimed that "the equation $x^n + y^n = z^n$ does not have positive integral solution (x, y, z) when n is an integer with $n \geq 3$." This claim is commonly known as "Fermat's last theorem". Despite efforts from many mathematicians, this theorem is only proven by Andrew Wiles in 1994, which was 329 years after Fermat's death.

   Now, we all know that "Fermat's last theorem" is true. Can you use it (or otherwise) to show that $\sqrt[n]{2}$ (the nth root of 2) is not a rational number for all integers $n \geq 3$?

6. (15%) [Adapted from a puzzle described in Tanya Khovanova's paper *Coins and Logic*]
   A city is divided into three regions, namely the Honest region, the Dishonest region, and the Alternating region. For some unknown traditions, if a person enters the Honest region, he or she must make only true statements. Similarly, if a person enters the Dishonest region, he or she must make only false statements. The most interesting thing is that, if a person enters the Alternating region, he or she must strictly make true and false statements, alternatingly (that is, if the current statement is true, the next statement must be false; while if the current statement is false, the next statement must be true).

   One night, the fire station received a call from a mobile phone: "Fire, help!" The operator couldn't identify the caller's location, so he asked, "Where are you calling from?" The reply was the Dishonest region. Assuming that the caller is not moving across the regions when he made the call, can we conclude if there was actually a fire going on? If so, can we be certain which region is the fire?

7. (Super Challenging: 10%)

   Three super smart boys are present in a room. Teacher Edward attaches a piece of paper on each of the boys' back, with an integer written on the paper. Each one can see the integers on the back of the other two boys, but not his own.

   Teacher Edward tells them that all the three integers are positive, and one of these integers is the sum of the other two.

   Now, Teacher Edward delivers a separate answer sheet to each of them, and asks them if they know what is the integer on their back. If so, write down that number on the answer sheet; else, write down "I don't know".

   Everyone answers.

   Teacher Edward says, "Good. So, everyone answers with 'I don't know'. How about now? If you know the integer on your back, write down the number on the answer sheet; else, write down 'I don't know'."

   Everyone answers.

   Teacher Edward says, "Good. So, everyone answers with 'I don't know' again. How about now? If you know the integer on your back, write down the number on the answer sheet; else, write down 'I don't know'."

   Everyone answers.

   Teacher Edward says, "Excellent. So, someone knows the integer on his back, and the integer is ... "

   (a) If the integer is 64, what are the other two integers?

   (b) If the integer is 70, what are the other two integers?

1. Identity Laws: $p \wedge T_0 \equiv p$ $\quad p \vee F_0 \equiv p$
2. Domination Laws: $p \wedge F_0 \equiv F_0$ $\quad p \vee T_0 \equiv T_0$
3. Idempotent Laws: $p \wedge p \equiv p$ $\quad p \vee p \equiv p$
4. Double Negation Law: $\neg(\neg p) \equiv p$
5. Commutative Laws: $p \wedge q \equiv q \wedge p$ $\quad p \vee q \equiv q \vee p$
6. Associative Laws: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ $\quad p \vee (q \vee r) \equiv (p \vee q) \vee r$
7. Distributive Laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8. De Morgan's Laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\quad \neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption Laws: $p \wedge (p \vee q) \equiv p$ $\quad p \vee (p \wedge q) \equiv p$
10. Negation Laws: $p \wedge \neg p \equiv F_0$ $\quad p \vee \neg p \equiv T_0$
11. De Morgan's Laws with Quantifiers: $\neg \forall x P(x) \equiv \exists x \neg P(x)$ $\quad \neg \exists x P(x) \equiv \forall x \neg P(x)$
12. Conditional Statement Equivalences: $p \to q \equiv \neg p \vee q$ $\quad p \to q \equiv \neg q \to \neg p$

Figure 1: Some useful logical equivalences

1. Modus Ponens:
   Premises: $p, \quad p \to q$ — Conclusion: q
2. Modus Tollens:
   Premises: $\neg q, \quad p \to q$ — Conclusion: $\neg p$
3. Hypothetical Syllogism:
   Premises: $p \to q, \quad q \to r$ — Conclusion: $p \to r$
4. Disjunctive Syllogism:
   Premises: $\neg p, \quad p \vee q$ — Conclusion: q
5. Addition:
   Premise: p — Conclusion: $p \vee q$
6. Simplification:
   Premise: $p \wedge q$ — Conclusion: p
7. Conjunction:
   Premises: $p, \quad q$ — Conclusion: $p \wedge q$
8. Resolution:
   Premises: $p \vee q, \quad \neg p \vee r$ — Conclusion: $q \vee r$
9. Universal Instantiation:
   Premise: $\forall x P(x)$ — Conclusion: $P(c)$, for any c
10. Universal Generalization:
    Premise: $P(c)$, for any c — Conclusion: $\forall x P(x)$
11. Existential Instantiation:
    Premise: $\exists x P(x)$ — Conclusion: $P(c)$, for some c
12. Existential Generalization:
    Premise: $P(c)$, for some c — Conclusion: $\exists x P(x)$

Figure 2: Some useful rules of inference