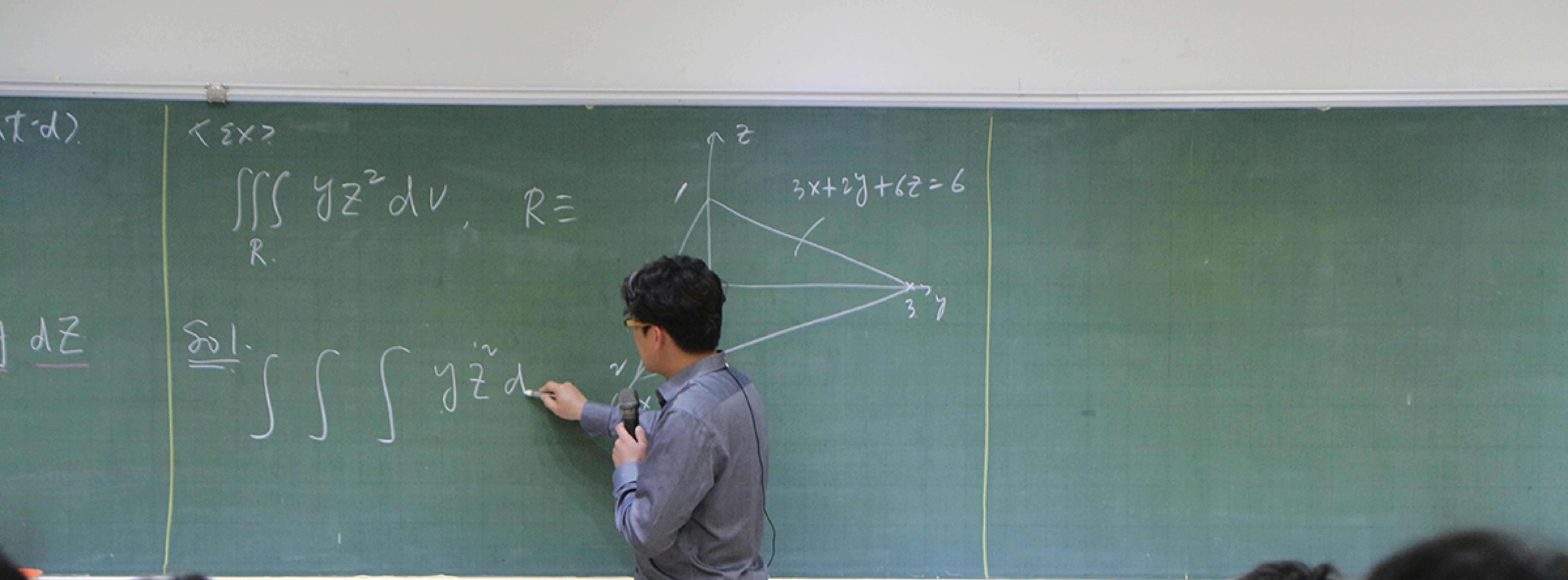
15.3. Double & Triple Integrals (cont'd)

II Triple Integral

$$\int_{z_1}^{z_2} \int_{y_1(z)}^{y_2(z)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\, \underline{dx}\, \underline{dy}\, \underline{dz}$$

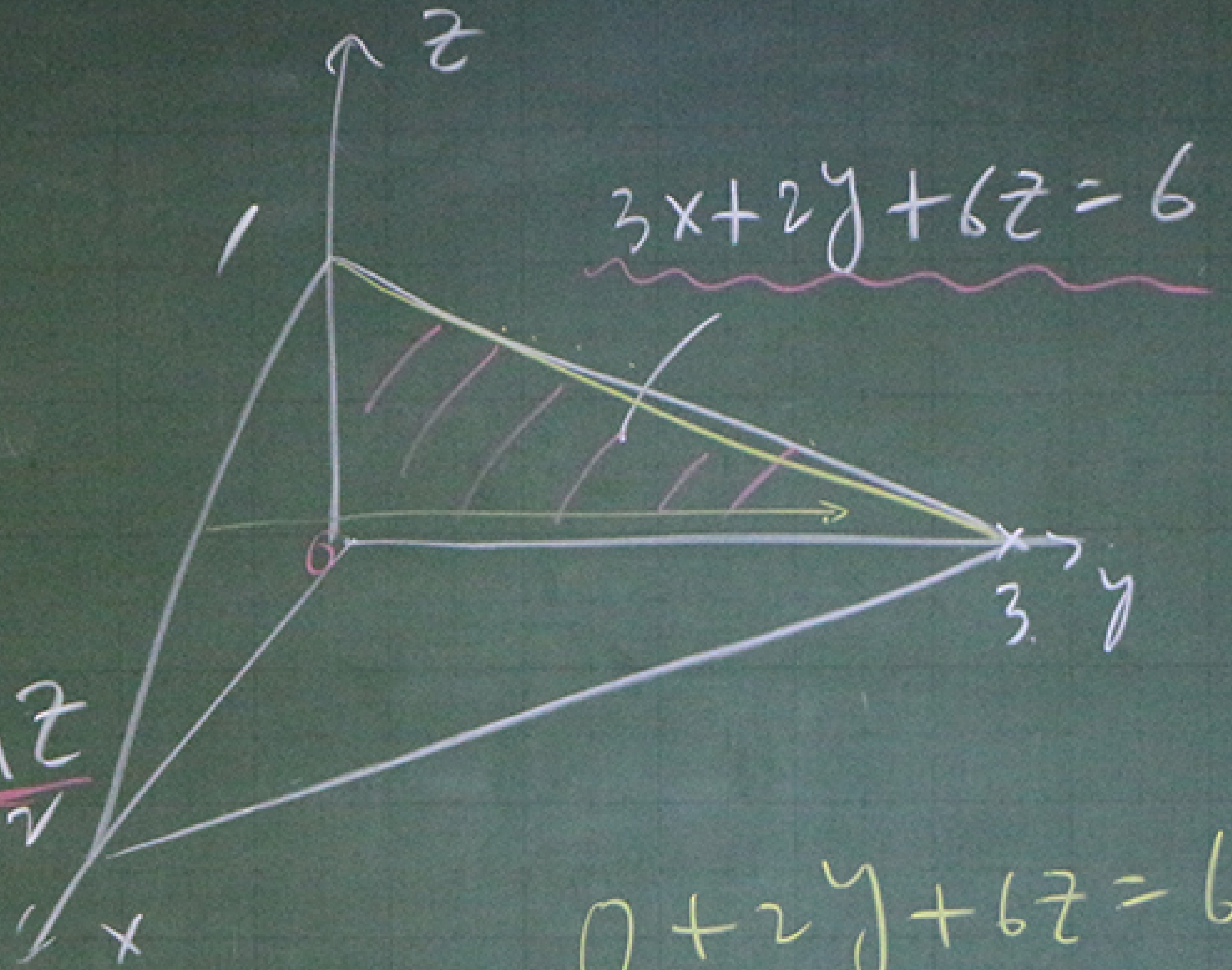

(t-d).
<Ex>
∭ yz² dV , R ≡
R.
z
3x+2y+6z=6
1
3
y
dZ
Sol.
∫ ∫ ∫ yz² d
2
x

<Ex>

$$\iiint_R yz^2\,dV, \quad R =$$

Sol.

$$\int_0^1 \int_0^{(3-3z)} \int_0^{\left(2-\frac{2y}{3}-z\right)} yz^2\,dx\,dy\,dz$$



$0+2y+6z=6$

$\rightarrow y = 3-3z$

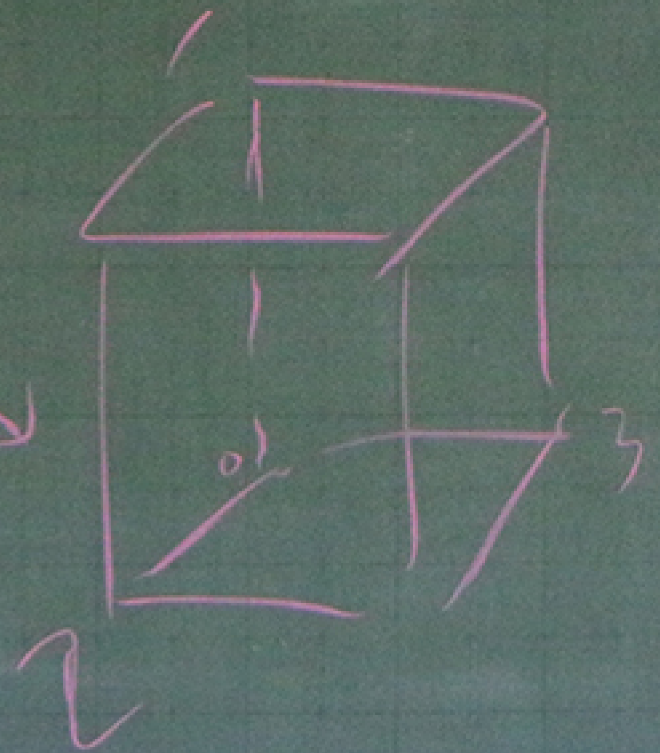
6

= 6

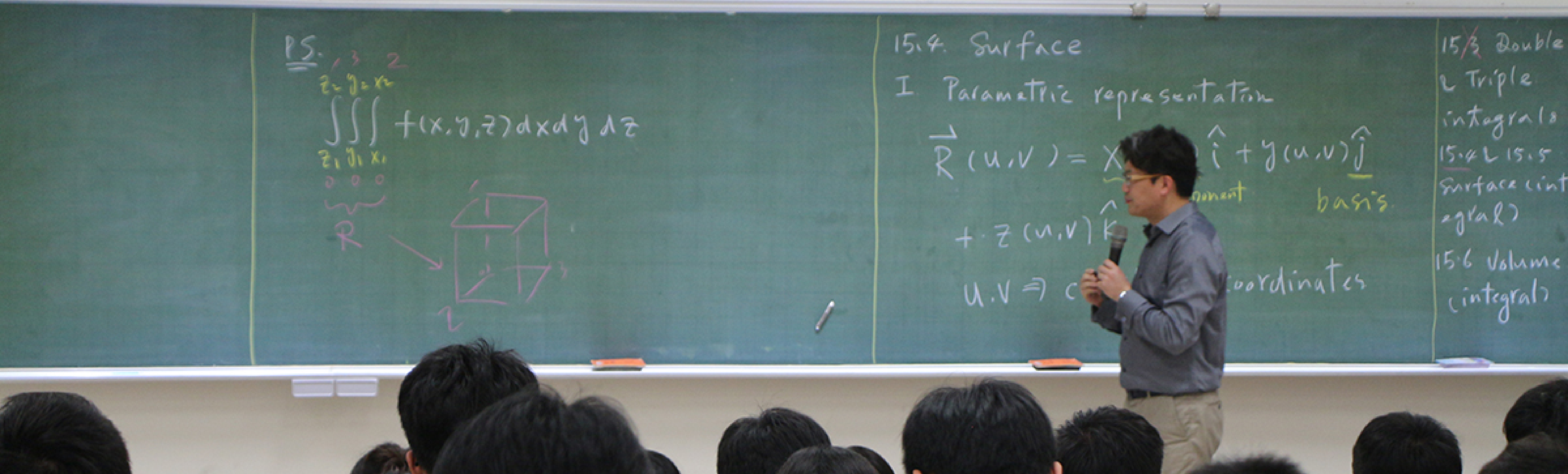
Z.

P.S.

$$\underbrace{\int_{\substack{z_1\\0}}^{\substack{1\\z_2}}\int_{\substack{y_1\\0}}^{\substack{3\\y_2}}\int_{\substack{x_1\\0}}^{\substack{2\\x_2}}}_{R} f(x,y,z)\,dx\,dy\,dz$$




15.4 Surface
I Parametric representation
15.3 Double
& Triple
integrals
15.4 & 15.5
Surface (int
egral)
15.6 Volume
(integral)


P.S.
15.4. Surface
I Paramatric representation
basis.
coordinates
15.6 Volume (integral)

15.4. Surface.

I. Parametric representation

$$\vec{R}(u,v) = \underbrace{x(u,v)}_{\text{component}}\,\hat{i} + y(u,v)\,\underbrace{\hat{j}}_{\text{basis}} + z(u,v)\,\hat{k}$$

$u, v \Rightarrow$ curvilinear coordinates.

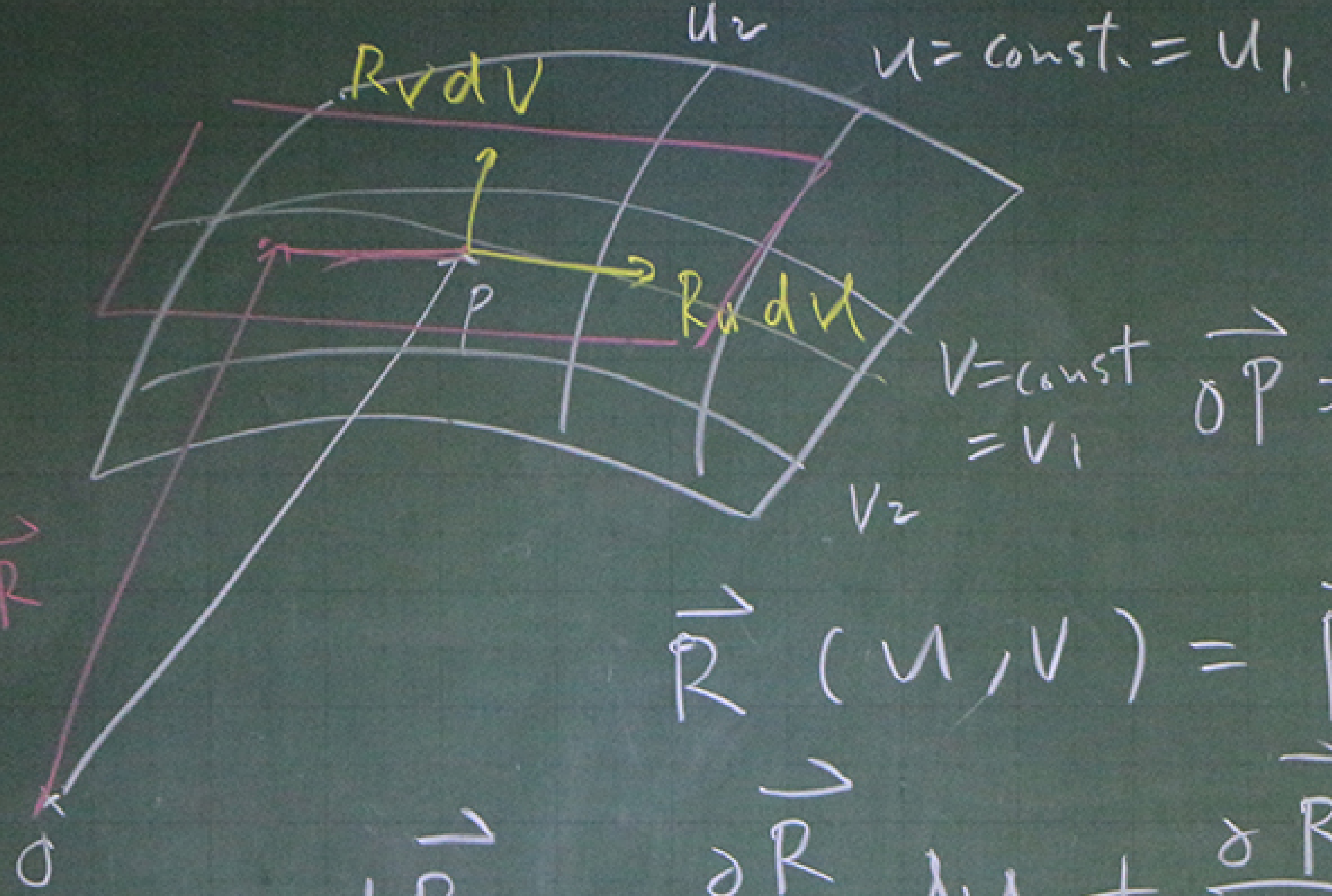
15.~~3~~ Double & Triple integrals

15.4 & 15.5 Surface (integral)

15.6 Volume (integral)


$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$
0 0 0
R
1
2
3


PS.
f(x, y, z) dx dy dz
R



$$\vec{OP} = \vec{R}_p(u, v)$$

$$\vec{R}(u, v) = \vec{R}(u(\tau), v(\tau))$$

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u} du + \frac{\partial \vec{R}}{\partial v} dv$$

$$= R_u du + R_v dv \quad \text{———} \quad (1)$$

$\hat{n}$ //

$(\vec{R}$

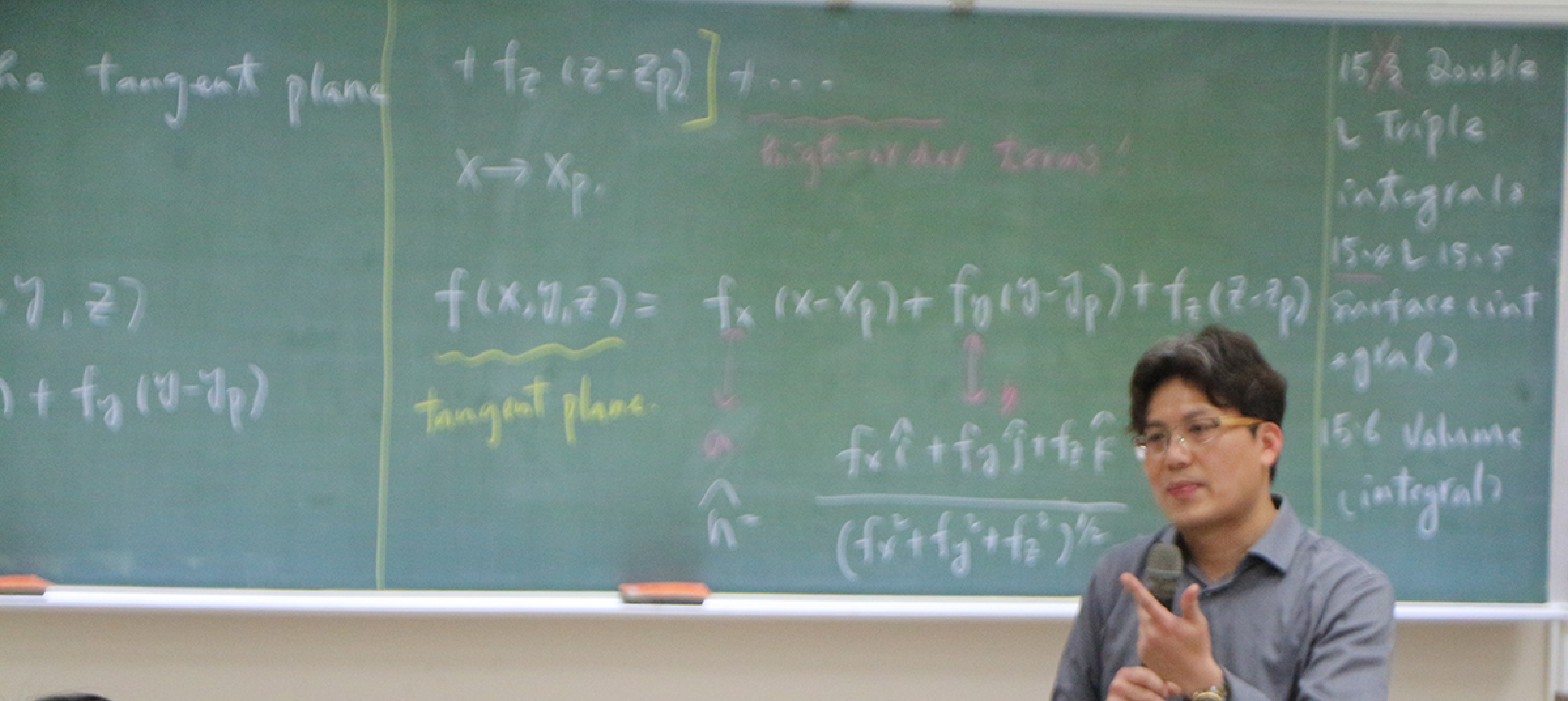
$a($

$$\hat{n} \parallel R_u \times R_v \rightarrow \hat{n} = \frac{R_u \times R_v}{|R_u \times R_v|} \quad \text{——— (2)}$$

$$(\vec{R} - \vec{R_p}) \cdot \hat{n} = 0 \quad ; \quad \hat{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$a(x - x_p) + b(y - y_p) + c(z - z_p) = 0$$

i.e., $ax + by + cz = d$ #. ——— (3)



he tangent plane
+ f_z (z - z_p)] + . . .
high-order terms!
x → x_p.
f(x, y, z) = f_x (x - x_p) + f_y (y - y_p) + f_z (z - z_p)
tangent plane
y, z)
) + f_y (y - y_p)
15.3 Double
& Triple
integrals
15.4 & 15.5
Surface (int
egral)
15.6 Volume
(integral)

II.

$f(x,y,z)=0$, find the tangent plane at P

TF of f at $P \to f(x,y,z)$

$$= \underbrace{f(x_p, y_p, z_p)}_{0} + \left[f_x (x-x_p) + f_y (y-y_p) \right.$$

ine

$$+ f_z(z - z_p)\Big] + \cdots$$

$\underbrace{}$ high-order terms!

$x \to x_p$,

$$\underbrace{f(x, y, z)}_{\text{tangent plane}} = f_x(x - x_p) + f_y(y - y_p) + f_z(z - z_p)$$

$$\begin{array}{ccc} \updownarrow & \updownarrow & \updownarrow \\ a & b & c \end{array}$$

$$\hat{n} = \frac{f_x \hat{i} + f_y \hat{j} + f_z \hat{k}}{(f_x^2 + f_y^2 + f_z^2)^{1/2}} \quad \#$$

~~15.3~~ Double & Triple integrals

~~15.4~~ & 15.5 surface (integral)

15.6 Volume (integral)

15.5 surface Integral

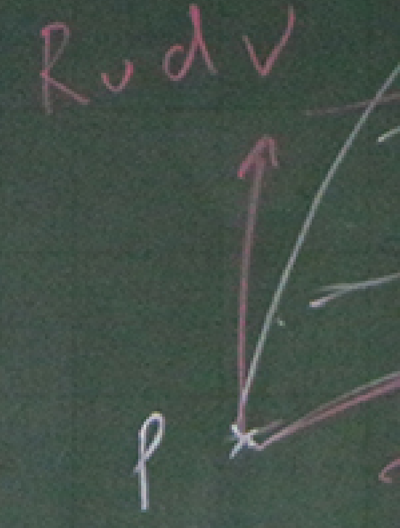
I

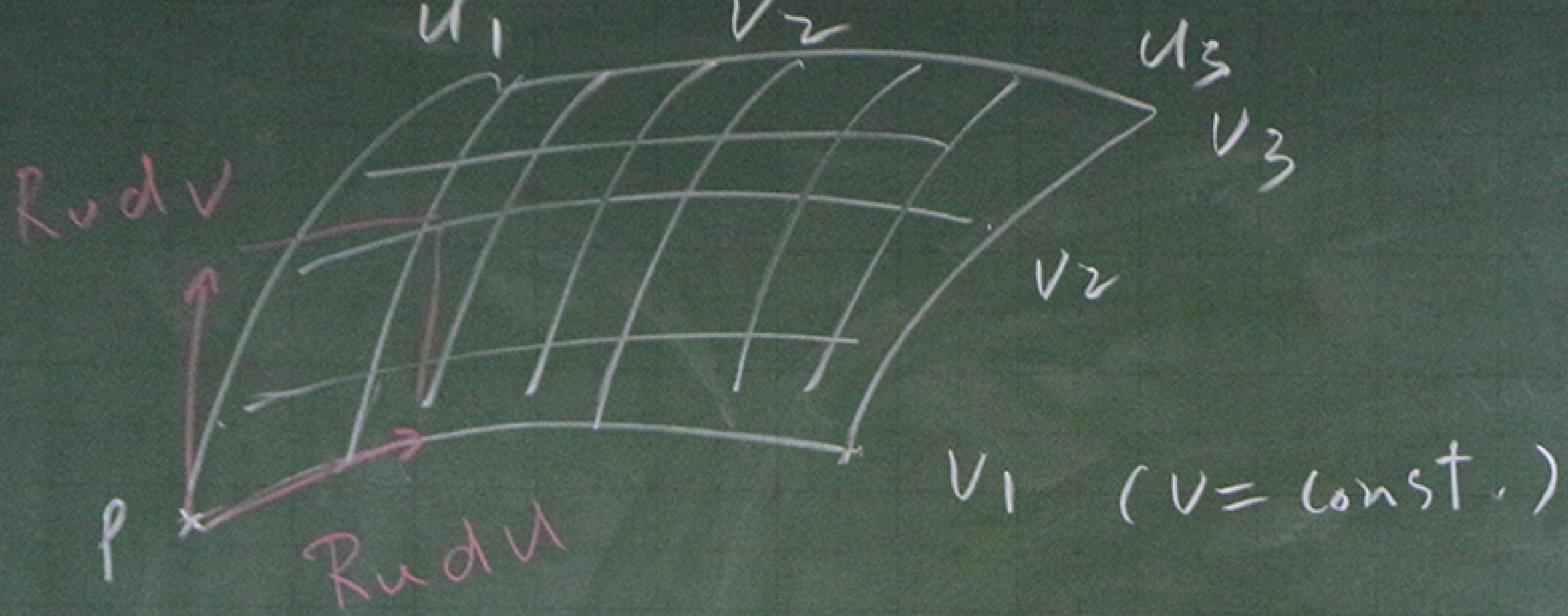
$$\iint f(x,y,z,t)\, dA$$

$$dA = dx\,dy \;;\; dy\,dx$$

"Cartesian coor."

$$dA = ?$$





$$dA = (R_u \times R_v)\, du\, dv \quad \#$$

$$\vec{R}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}.$$

$$R_u = x_u\,\hat{i} + y_u\,\hat{j} + z_u\,\hat{k}.$$

$$R_v = x_v\,\hat{i} + y_v\,\hat{j} + z_v\,\hat{k}$$

Method #1

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

Method # 2

$$R_u \times R_v = |R_u||R_v| \sin\theta$$

$$= |R_u||R_v| \sqrt{1 - \cos^2\theta}$$

$$= \left[|R_u|^2 |R_v|^2 - (R_u \cdot R_v)^2 \right]^{1/2}$$

15.~~3~~ Double & Triple integral

15.~~4~~ & 15.5 surface (integral)

15.6 Volume (integral)

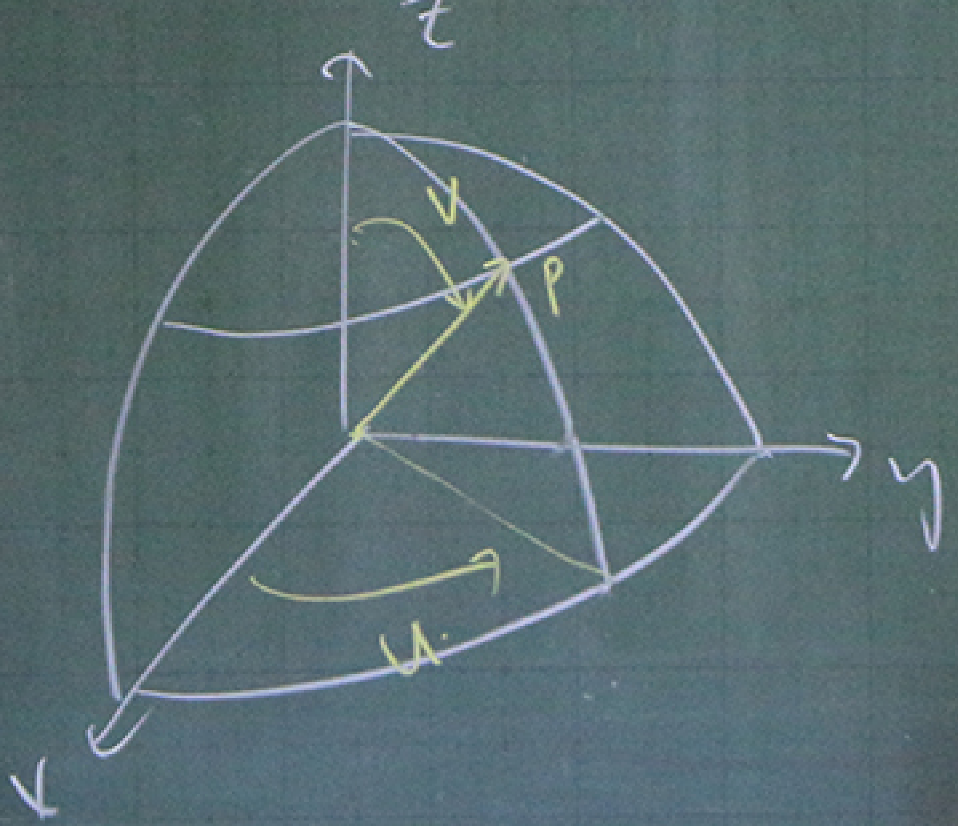
15.3 Double
& Triple
integrals
15.4 & 15.5
Surface (int
egral)
15.6 Volume
(integral)

15.3 Double
Triple
integrals
15.4 15.5
Surface (int
egral)
15.6 Volume
(integral)

<EX> Find the surface of a sphere w/ the radius a.

Sol

$$dA = |R_u \times R_v| \, du \, dv$$



$$x = a \sin v \cos u$$

$$y = a \sin v \sin u$$

$$z = a \cos v$$

re

u = longitude (経度)

v = latitude (緯度)

$$\vec{R} = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

$\hat{\ }$

u

$$\vec{R_u} = -a\sin v\sin u\,\hat{i} + a\sin v\cos u\,\hat{j}$$

$$\vec{R_v} = a\cos v\cos u\,\hat{i} + a\cos v\sin u\,\hat{j} - a\sin v\,\hat{k}$$

$$\vec{R_u}\times\vec{R_v} = \left[|\vec{R_u}|^2|\vec{R_v}|^2 - (\vec{R_u}\cdot\vec{R_v})^2\right]^{1/2}$$

$$= a^2\left[(\sin^2 v\sin^2 u + \sin^2 v\cos^2 u)(\cos^2 v\cos^2 u + \cos^2 v\sin^2 u + \sin^2 v) - 0\right]^{1/2}$$

$$= a$$

$$= a^2 \left[(\sin^2 V) \underbrace{(\cos^2 V + \sin^2 V)}_{1} \right]^{1/2}$$

$$= a^2 \sin V \ \#$$

$$8 \times \int_{\pi/2}^{\pi/2} \int_0^{\pi/2} a^2 \sin V \, du \, dV = 4\pi a^2$$

15.~~3~~ Double & Triple integrals

15.~~4~~ & 15.5 Surface (integral)

\# 15.6 Volume (integral)

Notes (x2).

N1. For 2D cases (i.e., $z = \text{const.}$)

$$\vec{R_u} \times \vec{R_v} = \Big[\underbrace{(x_u^2 + y_u^2)}_{|R_u|^2} \underbrace{(x_v^2 + y_v^2)}_{|R_v|^2} - (x_u x_v + y_u y_v)^2 \Big]^{1/2}$$

$$= \left| x_u y_v - x_v y_u \right| = \left| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right| = \frac{\partial(x, y)}{\partial(u, v)}$$

$$= J(u, v)$$

Namely, $|\vec{R}_u \times \vec{R}_v| \, du\, dv = J(u, v)\, du\, dv$

<Ex

Sn

<Ex.>.

Suppose that $u = r$, $V = \theta$

$$dA = J(r,\theta)\,dr\,d\theta = r\,dr\,d\theta \quad \#$$

$$dA_{ca.} = dx\,dy = dy\,dx$$

$$dA_p = r\,dr\,d\theta$$


<Ex.>
Suppose that u = r, v = θ
dA = J(r,θ) dr dθ = dr dθ #
dA_ca = dx dy
dx dy
dr dθ
dA_p = r dr dθ
N2
If z = f(x, y), such
x = u, y = v, z = f(x
R = u i + v j + f k
R_u = 1 i + 0 j + f_x k
R_v = 0 i + 1 j + f_y k

<u>NZ</u>

If $z = f(x, y)$, such that

$x = u$, $y = v$, $z = f(x, y)$, then

$$\begin{cases} \vec{R} = u\,\hat{i} + v\,\hat{j} + f\,\hat{k} \\ \vec{R_u} = 1\,\hat{i} + 0\,\hat{j} + f_x\,\hat{k} \\ \vec{R_v} = 0\,\hat{i} + 1\,\hat{j} + f_y\,\hat{k} \end{cases}$$

$\Rightarrow (\vec{R_u} \times \vec{R_v})\,du$

$AV =$

15.~~3~~ Double & Triple integrals

15.~~4~~ & 15.~~5~~ Surface (integral)

15.6 Volume (integral)

15. Volumn & Volumn Integral.

I. Volumn integral.

$$\iiint f(x, y, z, t)\, \underline{\underline{dV}} \quad ?$$

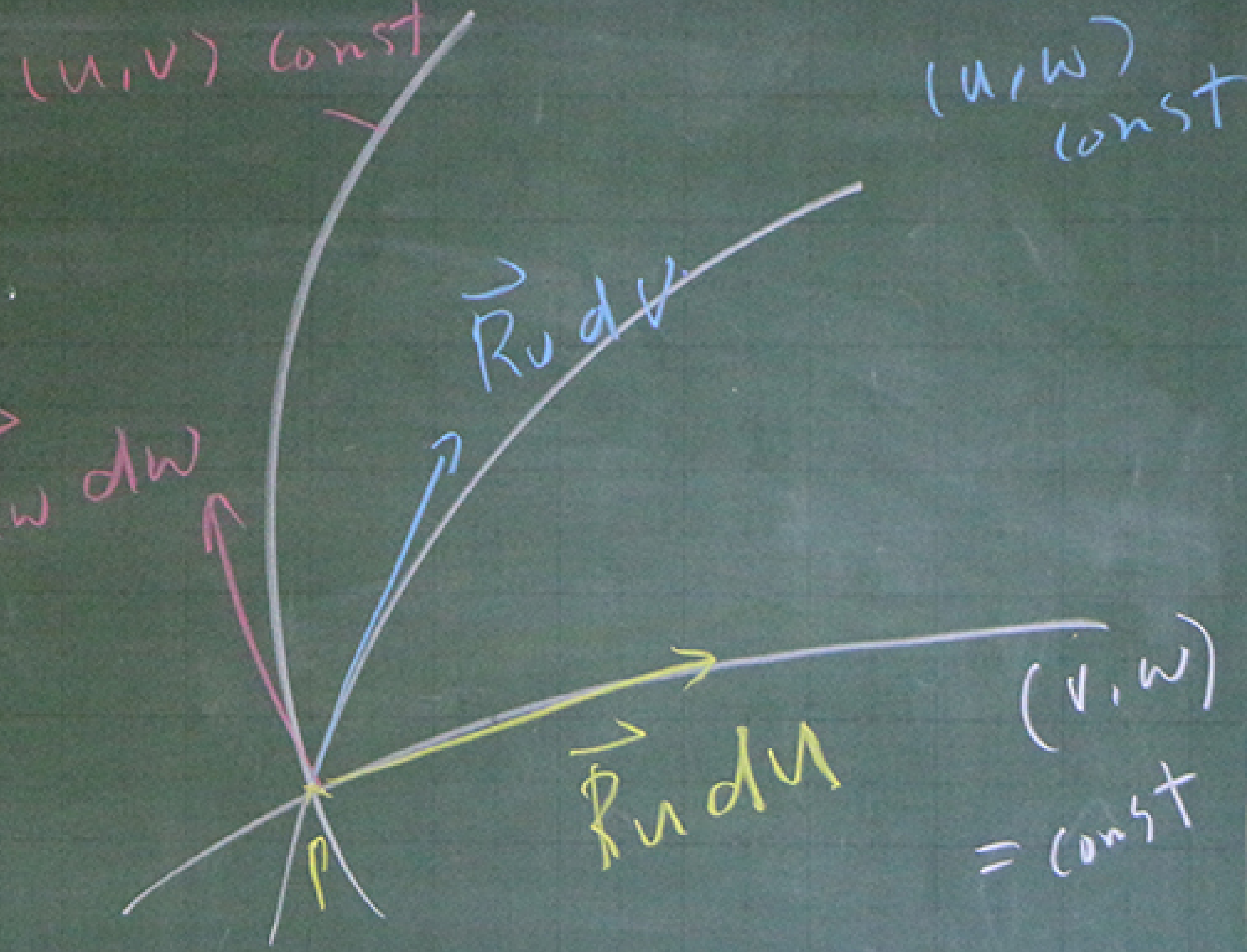
For Cartesian coordinates,

$$dV = dx\,dy\,dz.$$

$$\vec{R}(u,v,w) = x(u,v,w)\,\hat{i} + y(u,v,w)\,\hat{j}$$

$$+ z(u,v,w)\,\hat{k} \quad \#$$

$$dV = \left| \vec{R_u} \times \vec{R_v} \cdot \vec{R_w} \right| du\,dv\,dw$$



=

$$= \left\| \begin{matrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{matrix} \right\| \, du\,dv\,dw$$

absolute value.

determinant

$$= \left\| \begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix} \right\|$$

ust

w)

st

P.S.

P.S. $A \to A^T$ (T: transpose).

$\det A = \det A^T$.

$= J(u,v,w)\,du\,dv\,dw = dV$ #

15.~~3~~ Doub[le]
& Triple
integra[l]
15.~~4~~ & 15.~~5~~
Surface (
egral)
15.6 Volum[e]
(integral

* cylindrical coordinates

$$dV = J(r, \theta, z)\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz$$

* spherical coordinate

$$dV = J(\rho, \underset{\text{lat.}}{\underline{\underline{\phi}}}, \underset{\text{long.}}{\underline{\theta}})\, d\rho\, d\phi\, d\theta$$

$= \rho^2$

$$= \rho^2 |\sin\phi| \, d\rho \, d\phi \, d\theta$$

curves (1D) : $dS = \sqrt{r'(\tau)\cdot r'(\tau)}\, d\tau$

surfaces (2D) : $dA = |r_u \times r_v|\, du\, dv$

volumn (3D) : $dV = |r_u \times r_v \cdot r_w|\, du\, dv\, dw$

$= J(u, v, w)\, du\, dv\, dw$ #