13.4 Chain rule
(cont'd)

13.5 TF, mean value theorem

13.6 Implicit fxns. & Jacobi-an

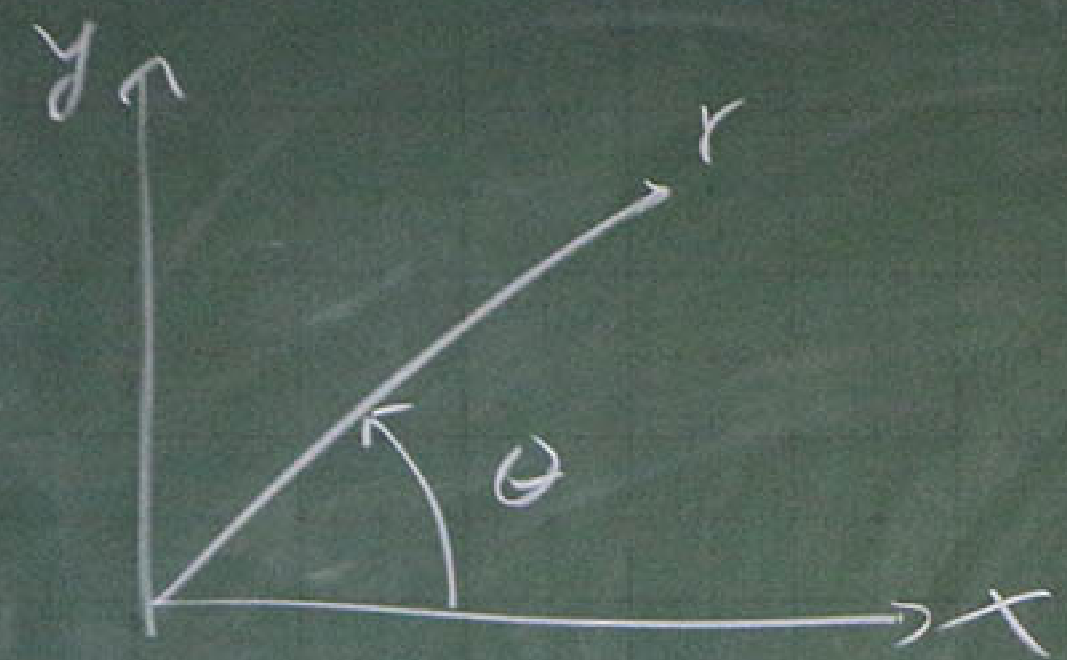
$$\begin{vmatrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_n} \\ \vdots & & \\ \dfrac{\partial f_n}{\partial u_n} & \cdots & \dfrac{\partial f_n}{\partial u_n} \end{vmatrix} = \frac{\partial (f_1, f_2 \cdots, f_n)}{\partial (u_1, \cdots, u_n)}$$

$$= J(u_1, u_2, \cdots, u_n) \neq 0 \;!!$$

Comment:

Jacobian not only identifies the unique existence of I.F., but also helps calculate partial derivatives!

<Ex.1> Polar coordinates.



$$f = x - r\cos\theta = 0$$

$$g = y - r\sin\theta = 0$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

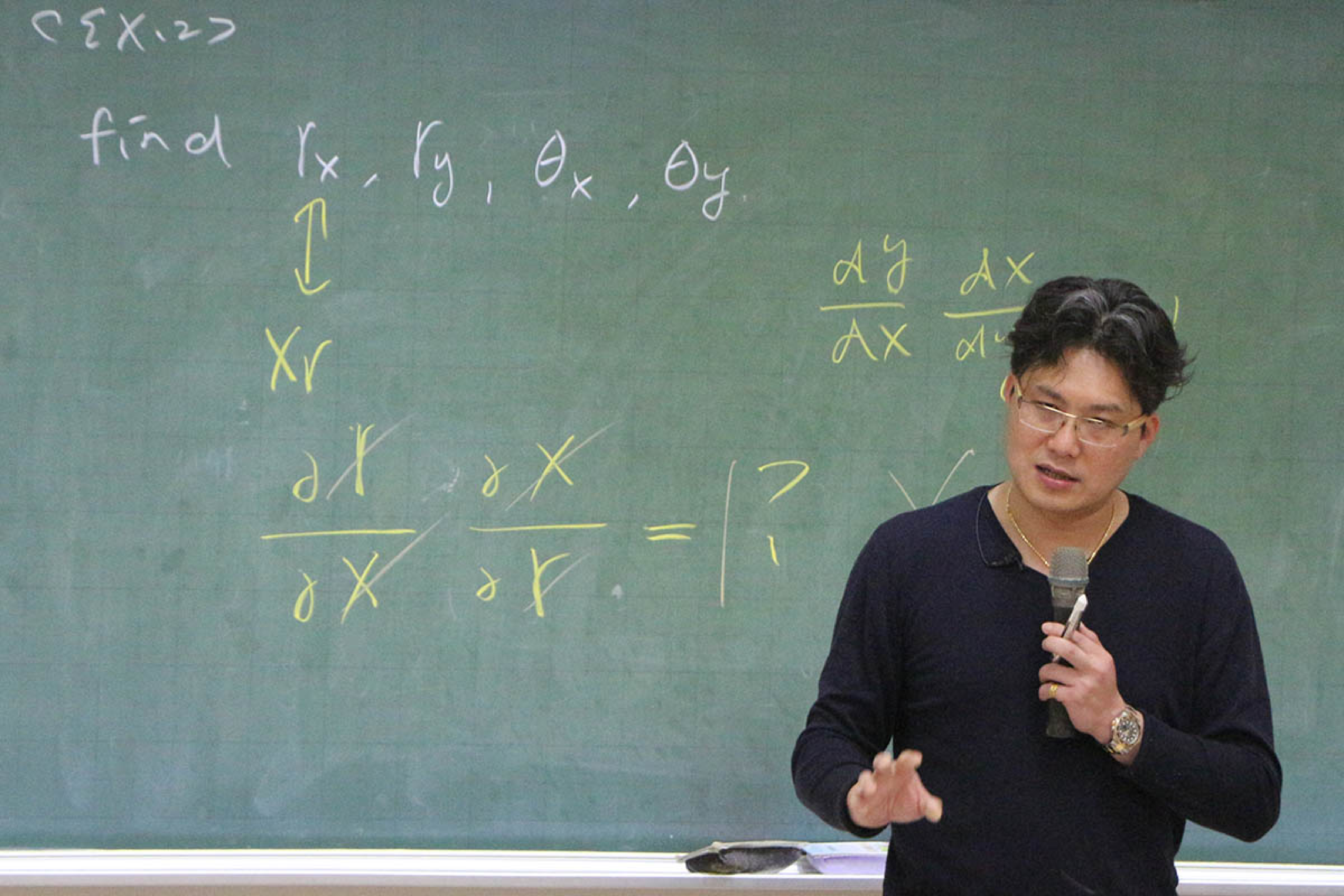
$$T(x, y) \rightarrow T(x(r,\theta), y(r,\theta))$$

$$\equiv t(r,\theta)$$

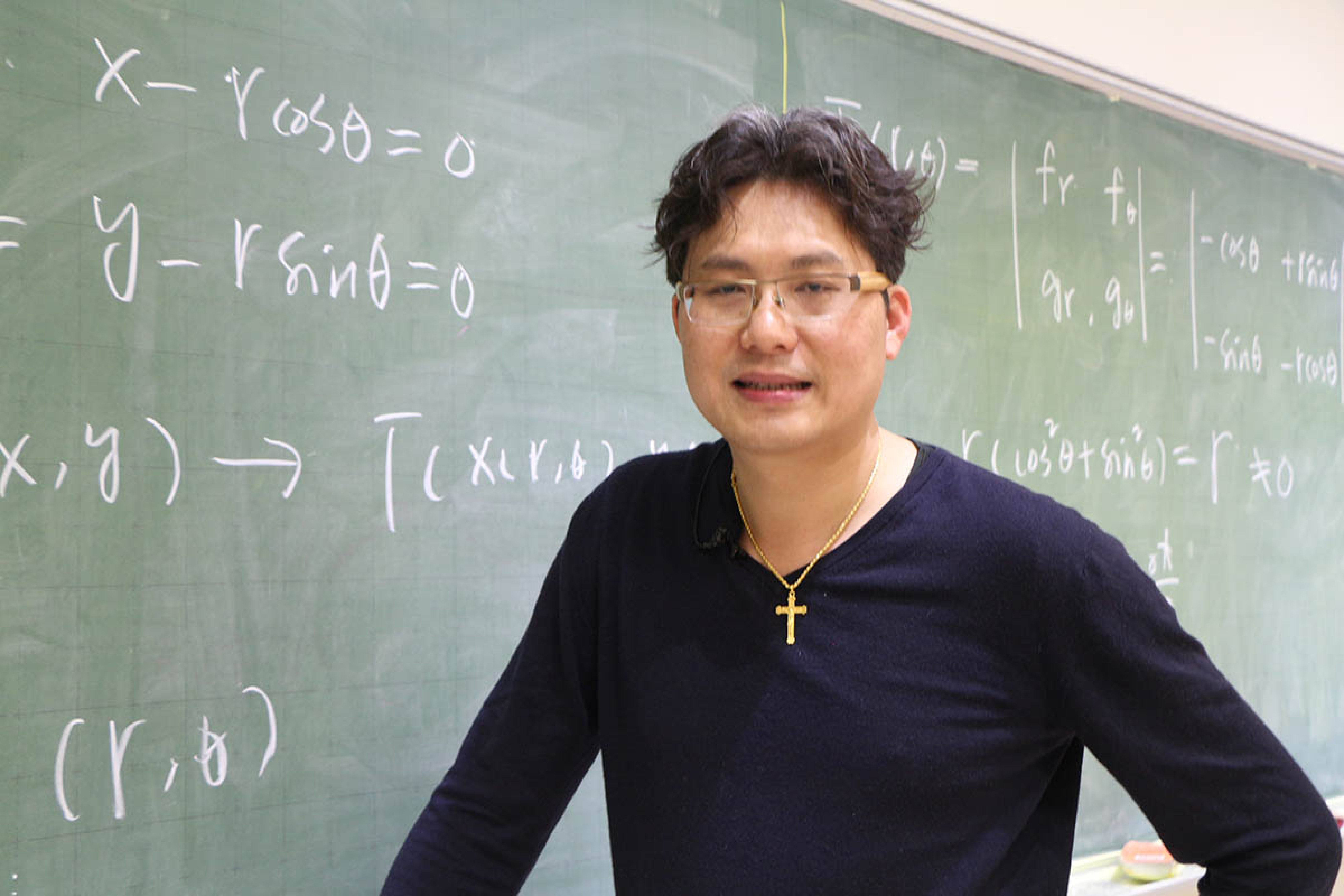
$$J(r,\theta) = \begin{vmatrix} f_r & f_\theta \\ g_r, & g_\theta \end{vmatrix} = \begin{vmatrix} -\cos\theta & +r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r \neq 0$$

2 $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \rightarrow \frac{\partial \lambda}{\partial r}, \frac{\partial \lambda}{\partial \theta}$



<EX.2>
find r_x, r_y, θ_x, θ_y
x_r
$\frac{\partial r}{\partial x} \frac{\partial x}{\partial r} = 1?$
$\frac{dy}{dx}$



$x - r\cos\theta = 0$
$y - r\sin\theta = 0$
$(x, y) \rightarrow$
(r, θ)
$r(\cos^2\theta + \sin^2\theta) = r \neq 0$

<Ex. 2>

Let $f(u,v) = uv^2$, where $u = 3x - y$, $v = x^2 y$

$f(u,v) = f(u(x,y), v(x,y)) \equiv \underbrace{F(x,y)}_{\text{composite fxn.}}$

Find $\partial F / \partial x$

Sol.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= \underline{\underline{v}}^2 \cdot 3 + 2\,\underline{\underline{uv}} \cdot 2xy$$

$$= 15x^4y^2 - 4x^3y^3 \quad \#$$

<comments, x2>

1. $\partial F/\partial x$ holds! ($\because$ y fixed)

2. $\dfrac{\partial F(\overset{\checkmark}{x}, \overset{\times}{y})}{\partial x}$, $\dfrac{\partial f(u, v, \overset{\checkmark}{x}, y)}{\partial x}$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\underline{\frac{\partial u}{\partial x}} + \frac{\partial f}{\partial v}\underline{\frac{\partial v}{\partial x}} + \frac{\partial f}{\partial x}\cancelto{1}{\frac{\partial x}{\partial x}} + \frac{\partial f}{\partial y}\cancelto{0}{\frac{\partial y}{\partial x}}$$

$$u(\underline{x}, \underline{y})$$

$$x$$

$$= f_u u_x + f_v v_x + f_x \quad \#$$

$$f_x$$

$$\cancel{\frac{\partial f}{\partial x}} = f_u u_x + f_v v_x + \cancel{f_x} \implies f_u u_x + f_v v_x =$$

$$\times$$

$$\frac{\partial G}{\partial x} = \underbrace{\frac{\partial g}{\partial x}}_{g_x}\cancelto{1}{\frac{\partial x}{\partial x}} + \frac{\partial g}{\partial y}\,\frac{\overset{\to 0}{\partial y}}{\partial x} + \underbrace{\frac{\partial g}{\partial u}}_{g_u}\underbrace{\frac{\partial u}{\partial x}}_{u_x} + \underbrace{\frac{\partial g}{\partial v}}_{g_v}\underbrace{\frac{\partial v}{\partial x}}_{v_x} = 0$$

$$g_u u_x + g_v v_x = -g_x \qquad \text{—————} \quad 2(b)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}\cancelto{0}{\frac{\partial x}{\partial y}} + \underbrace{\frac{\partial f}{\partial y}}_{f_y}\cancelto{1}{\frac{\partial y}{\partial y}} + \underbrace{\frac{\partial f}{\partial u}}_{f_u}\underbrace{\frac{\partial u}{\partial y}}_{u_y} + \underbrace{\frac{\partial f}{\partial v}}_{f_v}\underbrace{\frac{\partial v}{\partial y}}_{v_y}$$

$$f_u u_y + f_v v_y = -f_y$$

Likewise, you can compute $\dfrac{\partial G}{\partial x}$.

Now, let's compare $\partial(a,b)$

$$\begin{cases} f_u \underline{u_x} + f_v \underline{v_x} = -f_x \\ g_u \underline{u_x} + g_v \underline{v_x} = -g_x \end{cases} \qquad \partial(a,b)$$

PS

$AX = C$; $\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$x_j = \frac{\sum A_{ji} c_i}{\det A}$$ → co-factor expansion.

"Cramer's rule"

for Example

$$u_x = \frac{\begin{vmatrix} -f_x & f_v \\ -g_x & g_v \end{vmatrix} \; -J(x,v)}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} \equiv J(u,v) \; ; \quad v_x = \frac{\begin{vmatrix} f_u & -f_x \\ g_u & -g_x \end{vmatrix} \; -J(u,x)}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} \equiv J(u,v)$$

det A

< I.F theorem for multivariable casos >

$$\begin{cases} f_1(\underline{x_1, \cdots, x_n}; \underline{u_1, \cdots, u_n}) = 0 \\ \vdots \\ f_n(x_1, \cdots, x_n; u_1, \cdots, u_n) = 0 \end{cases}$$

$\begin{cases} 2 \text{ sets} \\ 1 \text{ set} \to n \text{ variab} \end{cases}$

$u_1(x_1, \cdots, x_n), u_2(x_1, \cdots, x_n)$

$\vdots$