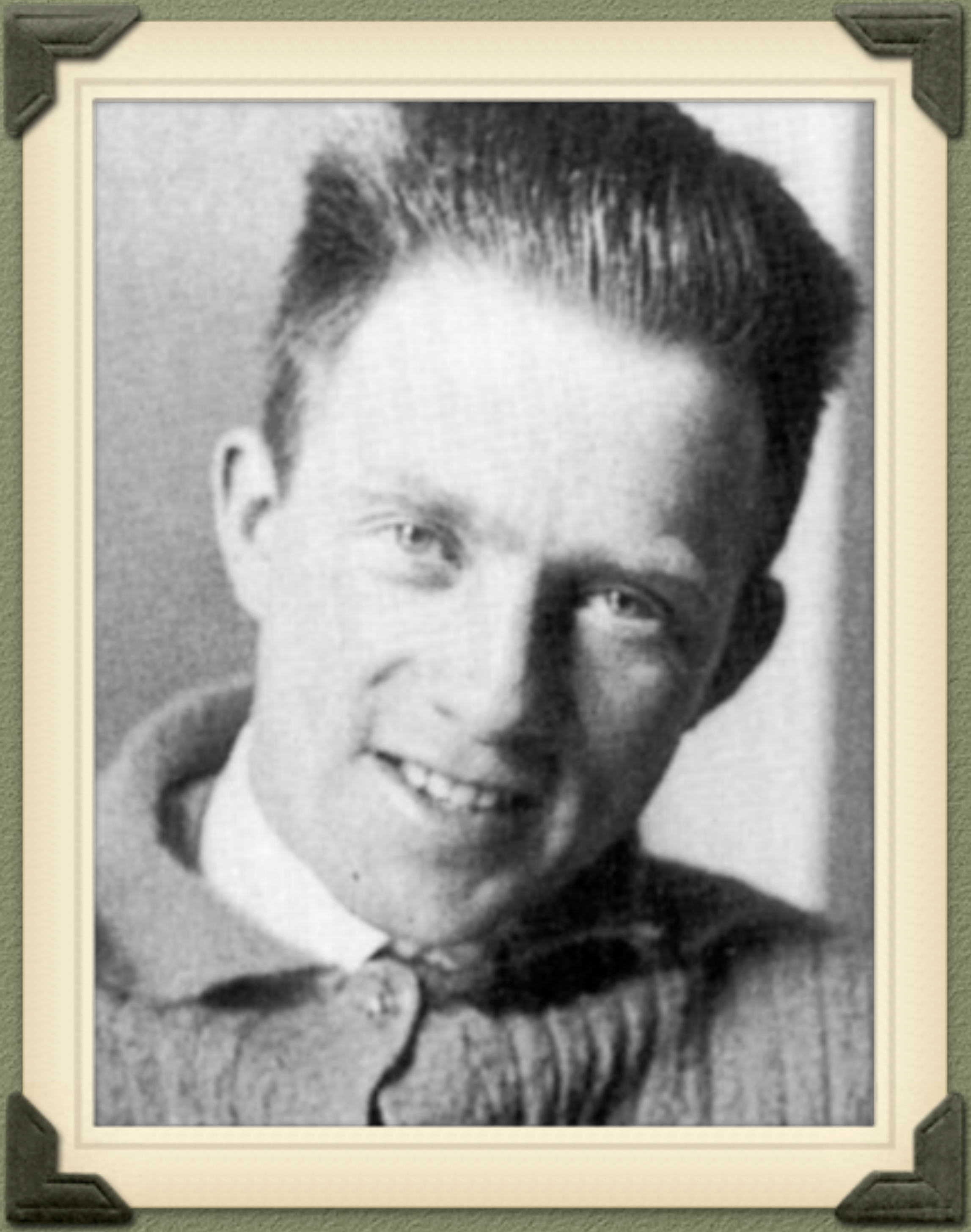
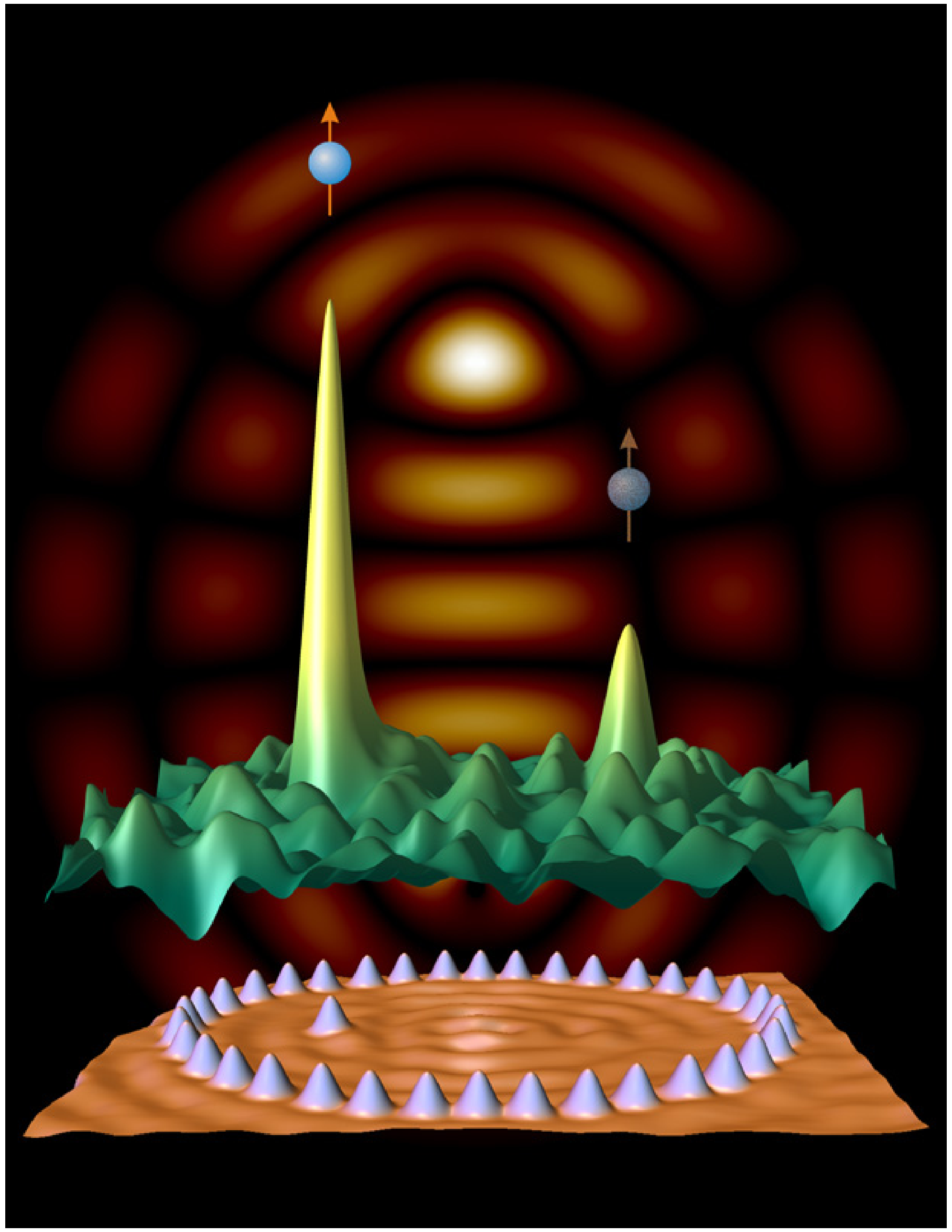
HH0132 - UNCERTAINTY PRINCIPLES



Quantum mirage

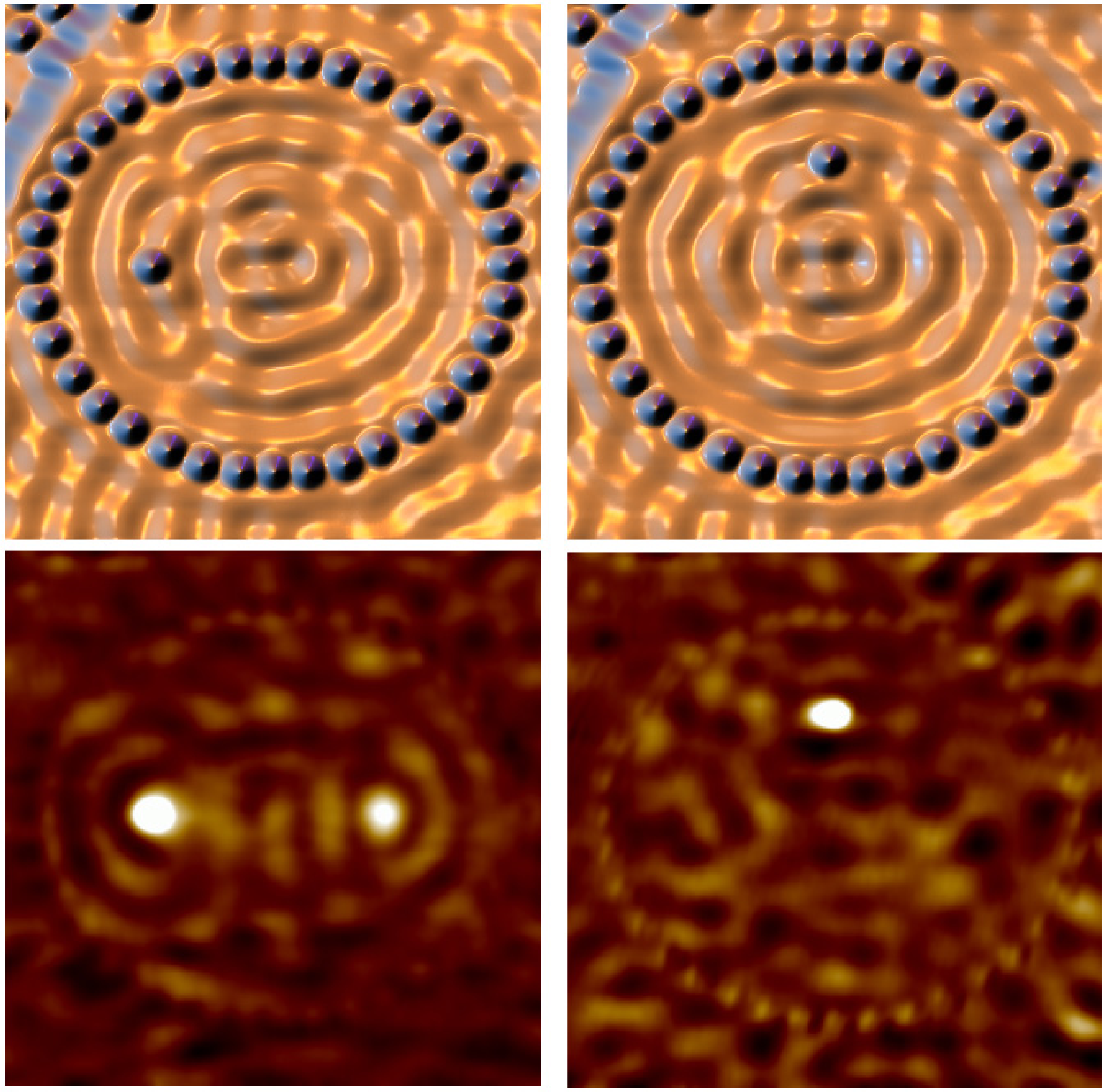


When placing one atom at the left focus, there is finite probability to find it at the right focus.

It is rather remarkable that quantum interferences lead to the exotic phenomena.

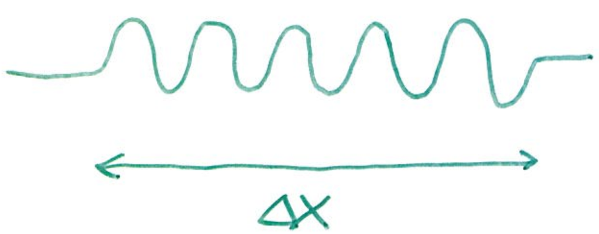
Off the focus

When the atom is placed away from the focus, the corresponding mirage at the other focus vanishes.



Uncertainty Principle.

Consider the following wave packet (e^- can be viewed as wave !!) with finite width Δx.



Count the number of wave front to determine k.

uncertain

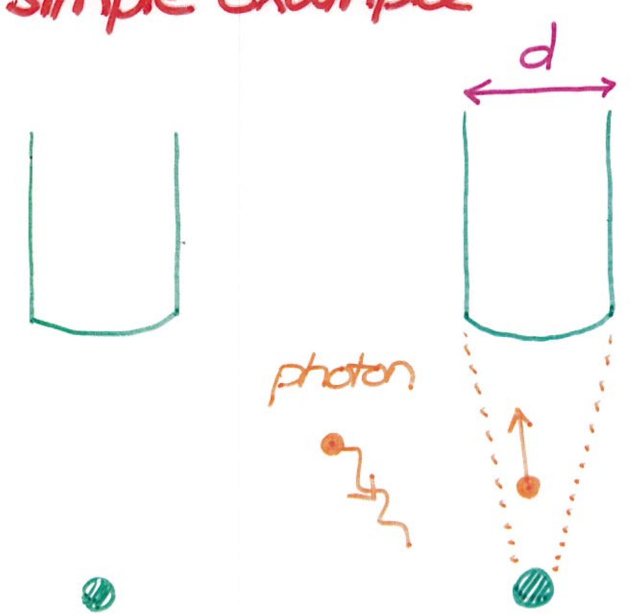
$$k \cdot \Delta x \cong 2\pi \, (N, N+1)$$

The uncertainty in k is $\Rightarrow$ $\Delta k \cong \dfrac{2\pi}{\Delta x}$

Thus, we arrive at the conclusion :

$$\boxed{\Delta x \, \Delta k \cong 2\pi} \xrightarrow{p=\hbar k} \boxed{\Delta x \, \Delta p \cong h}$$

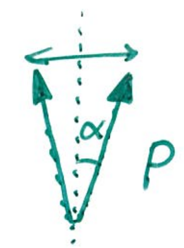
A simple example



$t=0$ — know p but not x

$t>0$ — $\Delta p \neq 0$, $\Delta x \neq 0$

The momentum uncertainty comes from interaction with photons.



$$\Delta p_e = \Delta p_{photon}$$

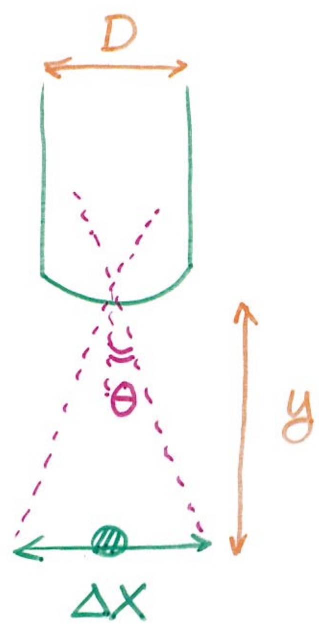
$$\Rightarrow \quad \Delta p_{photon} = p_{photon} \cdot \sin\alpha$$

$$\approx \left(\frac{h}{\lambda}\right)\left(\frac{d}{2y}\right) = \frac{hd}{2\lambda y}$$

Finally,

$$\boxed{\Delta p_e \approx \frac{hd}{2\lambda y}}$$

To reduce error, we want small $\underline{\underline{d}}$ and long $\underline{\underline{\lambda}}$!!



Since the microscope has limited resolution, one can not know the position precisely.

$$\Delta x \approx 2y\sin\theta \approx 2y\cdot\theta$$

Note that the minimum angle $\theta \approx \frac{\lambda}{d}$

$$\boxed{\Delta x \approx \frac{2y\lambda}{d}}$$

Collect results:

uncertainty principle ☺

$$\Delta p \approx \frac{hd}{2\lambda y}$$

$$\Delta x \approx \frac{2\lambda y}{d}$$

$$\Longrightarrow \quad \Delta x\,\Delta p \approx h \quad !!$$

More rigorous results for nuts

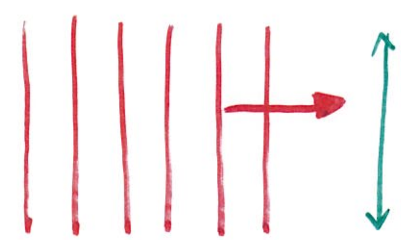
Previous arguments are nice. The crucial point for Heisenberg Uncertainty Principle to exist is the wave nature of matters.

$$\boxed{\begin{aligned} \Delta P_x\, \Delta x &\geq \tfrac{1}{2}\hbar \\ \Delta P_y\, \Delta y &\geq \tfrac{1}{2}\hbar \\ \Delta P_z\, \Delta z &\geq \tfrac{1}{2}\hbar \end{aligned}}$$

conjugate variables

☆ Note that only "conjugate variables" have uncertainty relations !!

i.e.



$\Delta y \to 0$ is possible.

definite P_x

$\Delta P_x = 0$

Another point of view: $\quad x P_x - P_x x = i\hbar$

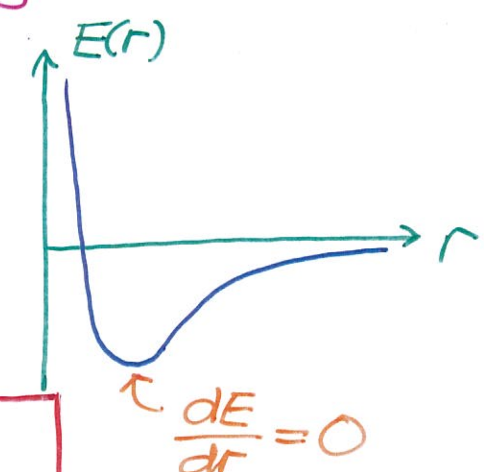
Size of H atom.

$\Delta x \approx r$

Use uncertainty principle to estimate the size of H atom. Assume $\Delta x \approx r$.

$$\Rightarrow \qquad \Delta p \approx p \approx \frac{\hbar}{r}$$

The total energy of electron is

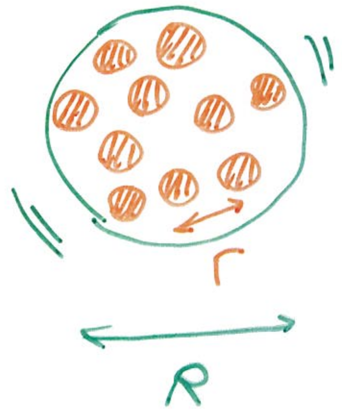


$$E = \frac{p^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0 r} \approx \frac{\hbar^2}{2m_e r^2} - \frac{Ze^2}{4\pi\epsilon_0 r}$$

Set $\frac{dE}{dr} = 0$ to determine r.

$$-\frac{\hbar^2}{m_e r^3} + \frac{Ze^2}{4\pi\epsilon_0 r^2} = 0 \quad \Rightarrow \quad \boxed{r = \frac{4\pi\epsilon_0 \hbar^2}{m_e Z e^2}}$$

The mass of star (I)



Consider a star with N particles, each of mass m.

$$E = \sqrt{m^2c^4 + p^2c^2} \approx pc$$ ← relativistic limit.

Making use of uncertainty relation $p \sim \frac{\hbar}{r}$

The kinetic energy K is roughly, $\boxed{K \approx pc \approx \frac{\hbar c}{r}}$

On the other hand, $V \sim -\frac{GMm}{R} = -\frac{GNm^2}{R} \longrightarrow -\frac{GN^{2/3}m^2}{r}$

Note that $\frac{4}{3}\pi R^3 \approx N\,\frac{4}{3}\pi r^3 \longrightarrow R = \sqrt[3]{N}\, r$

The total energy of the star is $\boxed{E = \frac{\hbar c}{r} - \frac{GN^{2/3}m^2}{r}}$

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The mass of star (II)

The total energy is $E = \frac{1}{r}\left(\hbar c - G N^{2/3} m^2\right)$

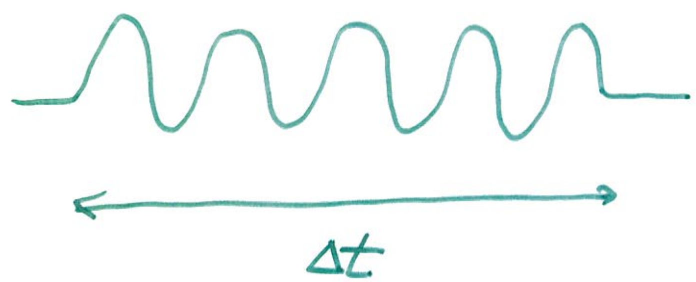
$$N_c = \left(\frac{\hbar c}{G m^2}\right)^{3/2} \approx 2 \times 10^{57} \sim 1.4\,\odot$$

For $N > N_c$, the total energy is always negative $\Rightarrow$ $r \downarrow$ indefinitely ..... the so-called gravitational collapse

the quantum uncertainty gives rise to some pressure to resist gravitational contraction !!

Life time and energy width.

Again, consider a train of signal with duration Δt. Now try to determine the angular frequency ω



$$\omega \Delta t \approx 2\pi\,(N, N+1)$$

Thus, the uncertainty in ω is $\boxed{\Delta\omega = \frac{2\pi}{\Delta t}}$

$$\Delta\omega\,\Delta t \approx 2\pi \implies (\hbar\Delta\omega)\,\Delta t \approx \hbar\cdot 2\pi$$

$$\Rightarrow \boxed{\Delta E\,\Delta t \approx h}$$

Δt : life time

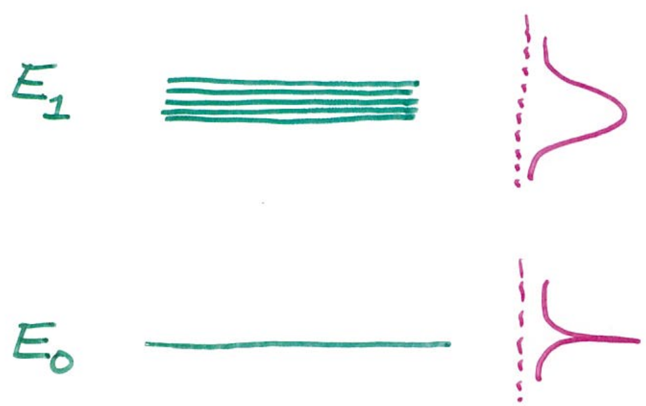
ΔE : energy width

Consider excited state in H atom

Since its life time Δt is finite, its energy can not be determined without uncertainty ΔE,

$$\boxed{\Delta E \approx \frac{\hbar}{\Delta t} \approx 10^{-7}\ eV}$$

Quite small



$\Delta E_1 \cdot \Delta t \approx \hbar$ finite life time.

$\Delta E_0 \approx 0$ life time $\rightarrow \infty$

Schrodinger equation

The dynamics of the wave function is described by the Schrodinger equation. Note that the time derivative is only first-order.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

wave function

To describe a quantum particle, we need the new concept of wave function:

$$\boxed{\psi(x,t)}$$

The interpretation of the wave function is the following: the square of the wave function is the probability density to find the particle at the location x and at the time t,

$$P(x,t) = |\psi(x,t)|^2 \,.$$

For instance, the probability to find the particle at the interval between a and b is the spatial integral of $|\psi(x,t)|^2$ in this regime.

stationary states

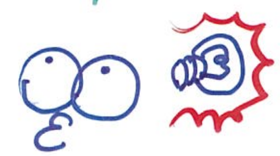
The time dependence of the stationary states can be solved rather easily. Make an educated guess of the following form:

$$\psi(x,t) = \phi(x) e^{-iEt/\hbar}$$

The time dependence of the Schrodinger equation drops out. This is the so-called time-independent Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) + V(x)\phi(x) = E\phi(x)$$

Wave Function ..... the mysterous beauty.

Q : Does the wave function $\psi(x,t)$ need to be complex ? 

OR, it's just a mathematical trick as derivations in diffractions, interferences .....

A : It depends.....

(1) stationary state without currents $\Rightarrow$ real ψ

(2) In general, ψ is complex ....

has interesting physical meaning.

Let's do some math .....

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \quad\Rightarrow\quad i\hbar\,\psi^*\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\psi^*\frac{\partial^2\psi}{\partial x^2} + V\psi^*\psi \quad (1)$$

Take complex conjugate

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^* \quad\Rightarrow\quad -i\hbar\,\psi\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\psi\frac{\partial^2\psi^*}{\partial x^2} + V\psi^*\psi \quad (2)$$

(1)-(2) gives the following equation.

$$i\hbar\left[\psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi\right] = -\frac{\hbar^2}{2m}\left[\psi^*\frac{\partial^2\psi}{\partial x} - \frac{\partial^2\psi^*}{\partial x^2}\psi\right]$$

$$i\hbar\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right]$$

ring the bell ?!

Finally, Finally....

$$\boxed{\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x}\left[\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)\right] = 0}$$

Probability density and current.

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x}\left[\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)\right] = 0 \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

This is continuity equation of some conserved quantity ☺

☆ What is it ?? conservation of probability !!

$$\begin{cases} \rho(x,t) = \psi^*\psi \\ j(x,t) = \dfrac{\hbar}{2mi}\left(\psi^*\dfrac{\partial\psi}{\partial x} - \dfrac{\partial\psi^*}{\partial x}\psi\right) \end{cases}$$

probability density and probability current

Note that if ψ is real, $\psi^* = \psi$. The probability current j is identically zero. From continuity eq, it implies $\rho(x,t) = \rho(x)$.

Simple system. $V(x)=0$ free particle.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

make an educated guess

$$\psi(x,t) = \phi_E(x)\, e^{-i\frac{E}{\hbar}t}$$

The time-dependent Schrödinger eq simplifies...

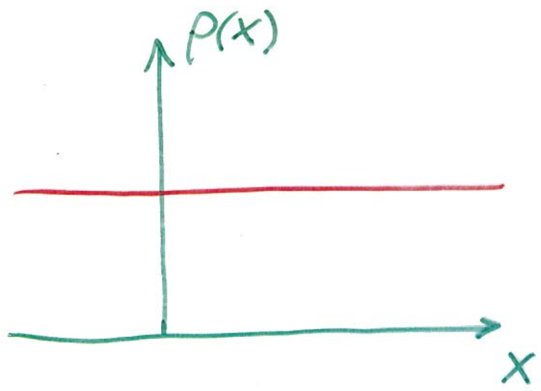
$$-\frac{\hbar^2}{2m}\frac{d^2\phi_E}{dx^2} = E\,\phi_E \quad \leftarrow \quad \phi_E(x) = e^{ikx}$$

Left: $-\frac{\hbar^2}{2m}(ik)^2\,\phi_E = \frac{p^2}{2m}\,\phi_E$ $\Rightarrow$ $\boxed{p = \pm\sqrt{2mE}}$

Right: $E\,\phi_E$

Finally, the plane wave solution is

$$\Longrightarrow \quad \boxed{\psi(x,t) = \frac{1}{\sqrt{L}}\, e^{i(kx-\omega t)}}$$



$\rho(x,t) = |\psi(x,t)|^2 = \frac{1}{\sqrt{L}}$ just a const !!

— We recover the relation $E = p^2/2m$ even in the quantum case.

— Probability density $\rho(x,t) = \frac{1}{\sqrt{L}}$ totally featureless

Q: How can we tell the momentum $p = \hbar k$?

— About Heisenberg's uncertainty principle....

$\Delta p = 0$ (definite momentum)

$\Delta x \to \infty$ (uniform distribution)

Good !!

Expectation value of physical observable.

The probability interpretation of $|\psi(x,t)|^2$ gives us the chance to compute the expectation value of observables:

$$\langle f(x)\rangle = \int_{-\infty}^{+\infty} dx\, f(x)\, P(x,t) = \int_{-\infty}^{+\infty} dx\, \psi^*(x,t)\, f(x)\, \psi(x,t)$$

t-dep comes from WF.

Now we can try to compute $\langle p\rangle$

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle = m\frac{d}{dt}\int dx\, \psi^*(x,t)\, x\, \psi(x,t)$$

$$= m\int dx \left[\frac{\partial\psi^*}{\partial t}\, x\, \psi + \psi^* x\, \frac{\partial\psi}{\partial t}\right](i\hbar)\frac{1}{(i\hbar)}$$

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} - V\psi \qquad\qquad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

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$$\langle p \rangle = \frac{\hbar}{2i}\int dx \left[\frac{\partial^2 \psi^*}{\partial x^2}\, x\, \psi - \underbrace{\psi^* x \frac{\partial^2 \psi}{\partial x^2}}\right]$$

The magic of integration by parts:

$$\int dx \frac{\partial^2 \psi^*}{\partial x^2}(x\psi) = \int x\psi\, d\left(\frac{\partial \psi^*}{\partial x}\right) = \cancel{x\psi \frac{\partial \psi^*}{\partial x}}^{\,0} - \int \frac{\partial \psi^*}{\partial x}\left(\psi + x\frac{\partial \psi}{\partial x}\right) dx$$

$$= -\psi^*\left(\psi + x\frac{\partial \psi}{\partial x}\right) + \int dx\ \psi^*\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} + x\frac{\partial^2 \psi}{\partial x^2}\right)$$

cancel the 2^{nd} term.

$$\langle p \rangle = \frac{\hbar}{2i}\int dx\ 2\ \psi^* \frac{\partial \psi}{\partial x} = \int dx\ \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi$$

It is tempting to assign $\boxed{\hat{p} = -i\hbar \frac{\partial}{\partial x}}$

The momentum can be viewed as differentiation on WF

Plane Wave again.

Let's calculated the averaged momentum $\langle p \rangle$.

$$\langle p \rangle = \int dx \; \psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x,t) \qquad \psi(x,t) = \frac{1}{\sqrt{L}} e^{i(kx-\omega t)}$$

$$= \hbar k \int dx \frac{1}{L} e^{-i(kx-\omega t)} e^{i(kx-\omega t)}$$

$$= \hbar k \Big(\underbrace{\frac{1}{L} \int dx}_{=1} \Big) \Rightarrow \boxed{\langle p \rangle = \hbar k}$$

It should be straightforward to see the average energy is

$$\langle E \rangle = \int dx \; \psi^*(x,t) \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x,t) = \frac{p^2}{2m} = E$$

Note that $j(x,t) = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) = \frac{\hbar}{2mi} \frac{1}{L} \cdot 2ik$

$$= \frac{1}{L} \cdot \left(\frac{\hbar k}{m} \right) = \rho \cdot v \quad \text{just like plane wave !!}$$

Commutator between $\hat{x}$ and $\hat{p}$.

$\hat{x} = x$

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$

(1) Suppose we measure $\hat{x}$ first, then measure $\hat{p}$ later on the wave fn Ψ

$$\hat{p}\hat{x}\Psi = -i\hbar\frac{\partial}{\partial x}\left[x\Psi(x)\right]$$

$$= -i\hbar\,\Psi(x) + x\left(-i\hbar\frac{\partial\Psi}{\partial x}\right)$$

(2) Suppose we measure $\hat{p}$ first, then measure $\hat{x}$ later.

$$\hat{x}\hat{p}\Psi = x\left(-i\hbar\frac{\partial\Psi}{\partial x}\right) \neq \hat{p}\hat{x}\Psi$$

origin of uncertainty principle.

(3) By comparison,

$$\hat{p}\hat{x}\Psi = -i\hbar\Psi + \hat{x}\hat{p}\,\Psi \Longrightarrow \boxed{\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar}$$

Stationary State.

Guess:

$$\psi(x,t) = \Phi_E(x)\, e^{-i\omega t}$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t)$$

The equation simplifies $\Rightarrow$ $\boxed{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right] \Phi_E(x) = E\, \Phi_E(x)}$

☆ $\rho(x,t) = |\psi(x,t)|^2 = |\Phi_E(x)|^2$ independent of time.

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial j}{\partial x} = 0$$

$j(x,t) = \text{const.}$

☆ all observables are time-independent !!

where the name comes from 😛

Q: What happens when adding two stationary states with energies E_1, E_2 ?

linear superposition at work again....

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\,\psi_1(x,t) + \frac{1}{\sqrt{2}}\,\psi_2(x,t) = \frac{1}{\sqrt{2}}\,\Phi_1(x)\,e^{-i\omega_1 t} + \frac{1}{\sqrt{2}}\,\Phi_2(x)\,e^{-i\omega_2 t}$$

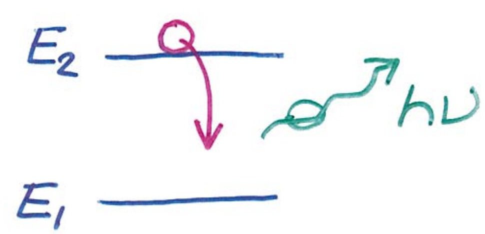
Let's compute the probability density

$$\rho(x,t) = |\Psi(x,t)|^2 = \frac{1}{\sqrt{2}}\left[\Phi_1^* e^{i\omega_1 t} + \Phi_2^* e^{i\omega_2 t}\right]\frac{1}{\sqrt{2}}\left[\Phi_1 e^{-i\omega_1 t} + \Phi_2 e^{-i\omega_2 t}\right]$$

$$= \frac{1}{2}\left(\Phi_1^*\Phi_1 + \Phi_2^*\Phi_2\right)$$ ← stationary part

$$+ \frac{1}{2}\Big[\underbrace{\Phi_2^*\Phi_1}_{=\,|\Phi_1||\Phi_2|e^{-i\delta}}\,e^{i(\omega_2-\omega_1)t} + \underbrace{\Phi_1^*\Phi_2}_{=\,|\Phi_1||\Phi_2|e^{i\delta}}\,e^{i(\omega_1-\omega_2)t}\Big]$$ ← oscillatory part.

$$\Rightarrow \boxed{|\Phi_1||\Phi_2|\,\cos(\omega_1 t - \omega_2 t + \delta)}$$



Choose $\psi_1(x,t) = \frac{1}{\sqrt{L}} e^{i(k_1 x - \omega_1 t)}$

$\psi_2(x,t) = \frac{1}{\sqrt{L}} e^{i(k_2 x - \omega_2 t)}$

$k_1 \neq k_2 \,,\quad \omega_1 \neq \omega_2$

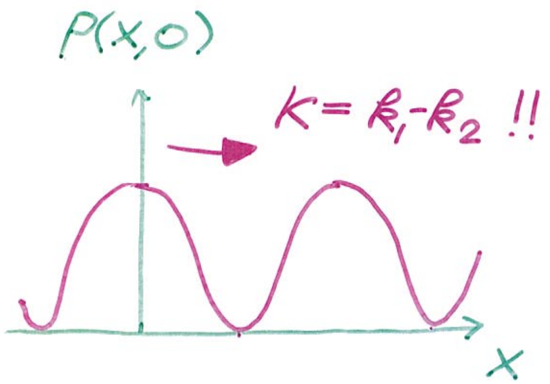
The probability density is $\rho(x,t) = \left| \frac{1}{\sqrt{2}} \psi_1 + \frac{1}{\sqrt{2}} \psi_2 \right|^2$

$$\rho(x,t) = \frac{1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2 \right) + |\Phi_1||\Phi_2| \cos(\Omega t + \delta)$$

$$= \frac{1}{L} + \frac{1}{L} \cos(\Omega t - kx)$$

$\Omega = \omega_1 - \omega_2$

$\delta = (k_2 - k_1)x$



The oscillating frequency is $\Omega = \omega_1 - \omega_2$ and the wave number is $k = k_1 - k_2$

Q: What happens when $\omega_1 = \omega_2$ but $k_1 \neq k_2$?

The End