- HH0129 -

Resolution Limit of Light

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When a distant light source passing through a circular aperture of diameter d, the image is not a point due to intrinsic diffraction. Thus, two distant sources can only be resolved when the angular separation is larger than θ_R defined as,

$$\theta_R = \sin^{-1}\left(\frac{1.22\lambda}{a}\right) \approx 1.22 \times \frac{\lambda}{a}. \tag{1}$$

The above criterion is referred as Rayleigh's criterion for resolving images. For a converging lens of diameter 50 mm shined by the visible light with wavelength $\lambda = 500$ nm, the Rayleigh's criterion gives

$$\theta_R \approx 1.22 \times \frac{\lambda}{d} \approx 10^{-5} \text{ rad}.$$

Because the angular separation θ_R is small, you may not notice the Rayleigh's criterion in noticeable way. However, it turns out to be the secret to appreciate the paintings of Georges Seurat as elaborated later.

• phasor technique

Let us revisit the double-slit interference. The electric fields of the waves at a particular point P vary with time,

$$E_1 = E_0 \sin\omega t, \qquad E_2 = E_0 \sin(\omega t + \phi),$$

where $\phi = 2\pi d \sin\theta/\lambda$ is the phase difference between the two waves. If the slit separation d is much smaller than the distance D to the screen, both electric fields are nearly parallel and their vector sum is just $E = E_1 + E_2$. In previous lecture, we show that the resultant electric field is

$$E = E_\theta \sin(\omega t + \varphi), \quad \text{where} \quad E_\theta = 2E_0 \cos(\phi/2), \tag{2}$$

and the phase shift of the resultant field is $\varphi = \phi/2$. The above expression gives a quantitative description of the intensity pattern for the double-slit

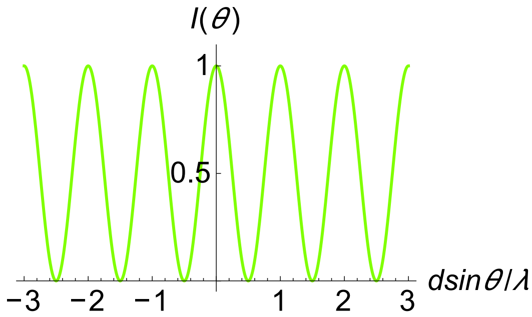

Figure 1: The intensity pattern for double-slit interference. The maximum intensity is set to unity in the figure.

interference. The intensity of the interference pattern is proportional to the square of the resultant electric field,

$$I_\theta = 4I_0 \cos^2 \alpha = 4I_0 \cos^2 \left(\frac{\pi d \sin\theta}{\lambda} \right), \tag{3}$$

as shown in Figure 1. The above result can also be derived by introducing a rotating vector called “phasor”. The first wave is represented by a phasor of magnitude E_0 rotating around the origin with angular frequency ω. The second wave is represented by the same magnitude E_0 but with a phase difference ϕ. It is quite interesting that the resultant electric field $E = E(t)$ is the sum of the projections of the two phasors on the vertical axis. Thus, the effect of interferences can be derived from the vector sum of the phasors. The phasor technique turns out to be more convenient when the number of light sources is large.

• diffraction of a narrow slit

When light passes though a narrow slit of width a comparable to the wavelength of the light λ, a series of alternating bright and dark bands are observed. The light source through the slit can be viewed as many tiny point sources and the resultant intensity on the screen can be calculated by the principle of linear superposition. The condition for destructive interference requires the tiny points sources can be separated into different zones with adjacent phase difference $\Delta\phi = \pi/2$,

$$a \sin\theta = n\lambda, \qquad n = \pm 1, \pm 2, \pm 3, \cdots . \tag{4}$$

With the geometric relation $\sin\theta \approx y/D$, one can figure out the location y_n of the n-th dark band on the projected screen at distance D.

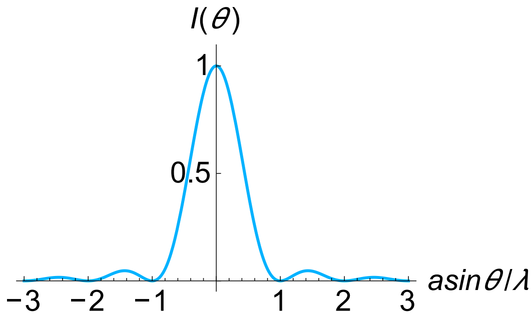

Figure 2: The intensity pattern for single-slit diffraction. The maximum intensity is set to unity in the figure.

What about the maxima of the diffraction pattern? Can we derive a quantitative description for the diffraction as we have done for the double-slit interference? The resultant intensity involves an infinite sum of a series of point sources with gradually changing phases. It can be done in principle but the algebra is rather messy. But, the phasor method comes to rescue. The series of tiny point sources corresponds to a chain of phasors forming a circular arc of the angle,

$$\Phi = 2\pi \frac{a \sin\theta}{\lambda}, \tag{5}$$

the maximum phase shift in the single slit. The resultant field E_θ is the chord of the arc. After some geometric analysis, the resultant field is

$$E_\theta = E_m \frac{\sin\alpha}{\alpha}, \quad \text{where} \quad \alpha = \frac{\Phi}{2} = \frac{\pi a \sin\theta}{\lambda}, \tag{6}$$

and E_m is the maximum amplitude of the electric field. The intensity I_θ for the diffraction pattern is proportional to the square of the electric field,

$$I_\theta = I_m \left(\frac{\sin\alpha}{\alpha}\right)^2 = I_m \left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2. \tag{7}$$

The intensity pattern is plotted in Figure 2. Note that the minima occurs when $\sin\alpha = 0$ (but $\alpha \neq 0$), equivalent to the condition of destructive interference $a\sin\theta = n\lambda$ derived before.

• Seurat's paintings

Consider two distant light sources passing through a single slit. Due to intrinsic diffraction of light, two series of bright-and-dark bands are formed on the screen. These two light sources are called "resolved" if the central maxima are separated further away from the *first* diffraction minimum,

$$\sin\theta = \frac{\lambda}{a}, \quad \rightarrow \quad \sin\theta = 1.22 \times \frac{\lambda}{a} \ \text{(circular aperture)}. \tag{8}$$

The choice of the first minimum is reasonable but certainly not absolute scientifically and the factor 1.22 is due to the circular geometry. This is Rayleigh's criterion for resolution limit.

In 1884-1886, Georges Seurat created the the painting *Sunday Afternoon on the Isle of La Grande Jatte* by the technique known as pointillism. The painting consists of closely spaced small dots (around 2 mm in diameter) of pure pigment. The diffraction of light passing through the observer's pupils induces the illusion of color mixing. Suppose the wavelength of the visible light is 550 nm and the diameter of the pupils is about 4 mm. The Rayleigh's criterion gives rise to a distance for colour mixing,

$$\theta_R \approx 1.22 \times \frac{550 \text{ nm}}{4 \text{ mm}} \approx 1.68 \times 10^{-4} \text{ rad} \approx 0.01 \text{ degree}. \tag{9}$$

The minimum distance for color-mixing illusion is $D = 2 \text{ mm}/\theta_R \approx 11.9$ m. So, to appreciate Seurat's paintings, one needs to keep an appropriate distance. The working principle of modern color monitor is the same as the pointillism. Amazing, isn't it?

• double-slit interference with diffraction

In analyzing the the double-slit interference, we assume that the slits are arbitrary narrow, i.e. $a \ll \lambda$. For such narrow slits, the central part of the screen is uniformly illuminated by the diffracted waves from each slit. In this narrow-slit limit, the interference pattern exhibits uniform intensity peaks as shown in Figure 1.

In practice, the narrow-slit condition $a \ll \lambda$ is usually not met for visible light. Thus, the effect of diffraction can not be ignored anymore. The resultant electric field is the sum of two infinite series of phasors from the two slits. Let us carry out the sum for each slit first,

$$\begin{aligned} E(t) &= E_1(t) + E_2(t) \\ &= E_d \sin(\omega t + \alpha) + E_d \sin(\omega t + \alpha + \phi) \\ &= E_m \left(\frac{\sin\alpha}{\alpha}\right) \Big[\sin(\omega t + \alpha) + \sin(\omega t + \alpha + \phi) \Big]. \end{aligned} \tag{10}$$

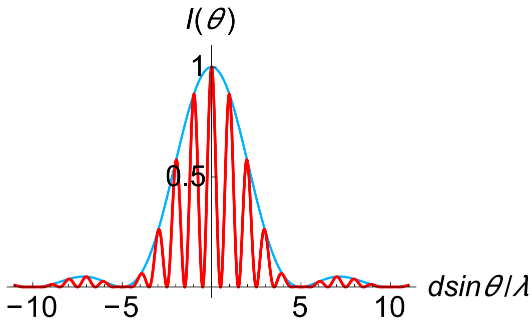

Figure 3: The intensity pattern for double-slit interference with finite width $a = 0.2d$. The maximum intensity is set to unity in the figure.

Notice that the above form is exactly the same as the double-slit interference except the amplitude $E_d = \sin(\pi a \sin\theta/\lambda)/(\pi a \sin\theta/\lambda)$ now depends on the diffraction angle. Following the same analysis, the resultant amplitude of the electric field is

$$E_\theta = 2E_d \cos(\phi/2) = 2E_m \cos(\phi/2)\left(\frac{\sin\alpha}{\alpha}\right). \tag{11}$$

Thus, the intensity pattern for the double-slit interference with diffraction included takes the following form,

$$I_\theta = I_m \cos^2(\phi/2)\left(\frac{\sin\alpha}{\alpha}\right)^2, \tag{12}$$

where $\phi = 2\pi d \sin\theta/\lambda$ is the phase difference due to the geometric separation of the two slits and $\alpha = \pi a \sin\theta/\lambda$ arises from diffraction within each slit. The intensity plot is shown in Figure 3. The envelope of the intensity pattern for the double-slit interference with finite width is the diffraction profile.

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