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Why Can Light Propagate in Vacuum?

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Sunlight is vital for most creatures in this warm and humid planet. From interference experiments, scientists were once convinced that light is some fast-travelling waves. But, as mechanical waves need media to propagate, how can light propagate in the vast vacuum between the Sun and the Earth? Let us review the wave equation in one dimension studied last semester,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u = u(x, t)$ is the wave function and v is the wave speed. In three dimensions, the wave function picks up all spatial dependence, $u = u(x, y, z, t)$, and satisfies the following wave equation,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u, \tag{2}$$

where the Laplacian ∇^2 is a shorthand notation for the sum of all second derivatives in each dimension,

$$\nabla^2 \phi \equiv \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}. \tag{3}$$

Despite of its formidable appearance, the wave equation is nothing but Newton's second law applying to molecular dynamics at microscopic scale. That is exactly why mechanical waves need media to propagate. Without the molecular motions in the media, its collective behaviors, i.e. waves, disappear altogether.

• Maxwell equations in vacuum

But, what about light? Does Newton's second law reign over the light propagation? Nope, the dynamics of the electric and magnetic fields is described by the Maxwell equations. In vacuum, there is no source term (charge density ρ and/or current density $\boldsymbol{J}$). It is quite remarkable that the Maxwell equations in vacuum still hosts non-trivial solutions with vivid dynamics,

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{4}$$

where c is the speed of light in vacuum. From the above equations, we can show that the electric and magnetic fields satisfy the wave equation. In addition, the Maxwell equations tell us that the electromagnetic waves are transverse (with respect to the propagation directions) with two distinct polarizations.

• wave equation for light

To derive the wave equation for light, it is helpful to prove the following identity[1],

$$\nabla \times (\nabla \times \boldsymbol{X}) = \nabla(\nabla \cdot \boldsymbol{X}) - \nabla^2 \boldsymbol{X}. \tag{5}$$

In Cartesian coordinates, just compute the curl operations twice and the identity follows. Focus on one of the Maxwell equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{6}$$

Taking the curl on both sides and exchanging the order of temporal and spatial derivatives, it leads to

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}). \tag{7}$$

The magnetic field on the right-hand side can be eliminated by one of the Maxwell equations,

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{8}$$

Note that the above equation contains solely the electric field now. Making use of the identity shown before, $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$. Because $\nabla \cdot \boldsymbol{E} = 0$ in vacuum, only the second terms survives,

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E} \quad \text{(in vacuum)}. \tag{9}$$

[1]Compare the identity to the usual one for vectors, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$. It seems that one can treat the differential operator ∇ as a vector as long as the order is carefully arranged. Why?

In consequence, the electric field satisfies the following wave equation,

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{10}$$

Repeating similar steps for the magnetic field, it is straightforward to show that the same wave equation emerges,

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}. \tag{11}$$

From the above wave equations, we realize that both the electric and the magnetic fields can propagate in vacuum without the presence of any source term! Besides, the wave equations for light is linear and the principle of linear superposition works. Any complicated shape can be decomposed into simple sinusoidal waves so that we only need to study these simple solutions.

• electromagnetic waves

The simplest sinusoidal wave in three dimensions is the plane wave,

$$\boldsymbol{E}(x,y,z,t) = \boldsymbol{e}\ \sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi_e\right), \tag{12}$$

$$\boldsymbol{B}(x,y,z,t) = \boldsymbol{b}\ \sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi_b\right). \tag{13}$$

Here $\boldsymbol{k}$ is the wave number (a vector) and $\omega = kc$ is the angular frequency. Because we are dealing with vector fields, their amplitudes $\boldsymbol{e}$ and $\boldsymbol{b}$ are vectors as well. The phases for the electric and the magnetic fields are denoted as ϕ_e and ϕ_b respectively. We shall soon find all relations between these parameters in the following. First of all, let us compute the divergence of the electric field in the plane wave,

$$\nabla\cdot\boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y}. \tag{14}$$

The first term (corresponding to the x-direction) gives the contribution,

$$\frac{\partial E_x}{\partial x} = k_x e_x\ \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi_e). \tag{15}$$

Adding all contributions together, the divergence of the electric field is

$$\nabla\cdot\boldsymbol{E} = (\boldsymbol{k}\cdot\boldsymbol{e})\ \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi_e). \tag{16}$$

The Maxwell equation $\nabla \cdot \boldsymbol{E} = 0$ leads to the geometric constraint on the amplitude of the electric field,

$$\boldsymbol{k} \cdot \boldsymbol{e} = 0. \tag{17}$$

Similarly, the Maxwell equation $\nabla \cdot \boldsymbol{B} = 0$ leads to the geometric constraint on the amplitude of the magnetic field,

$$\boldsymbol{k} \cdot \boldsymbol{b} = 0. \tag{18}$$

So, both amplitudes are perpendicular to the propagating direction and render the electric and magnetic waves transverse.

But, this is not the end of the story – the Maxwell equations tell us more. Turning our attention to the curl of the fields,

$$\nabla \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{\boldsymbol{z}}. \tag{19}$$

Working out the derivatives for the x-component, it gives

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = (k_y e_z - k_z e_y)\ \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \phi_e) \tag{20}$$

Repeating the same calculations for the other components, the result can be cast into the following form,

$$\nabla \times \boldsymbol{E} = (\boldsymbol{k} \times \boldsymbol{e})\ \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \phi_e). \tag{21}$$

Substituting into the Maxwell equation relating the electric and the magnetic fields, it requires that the phases are the same $\phi_e = \phi_b$ and puts a geometric constraint on the amplitudes,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \rightarrow \quad \boldsymbol{k} \times \boldsymbol{e} = \omega \boldsymbol{b}. \tag{22}$$

The dispersion relation $\omega = kc$ simplifies the relation a bit,

$$\hat{\boldsymbol{k}} \times \boldsymbol{e} = c\, \boldsymbol{b}. \tag{23}$$

Similarly, starting from the other Maxwell equation relating the electric and the magnetic fields, it again requires $\phi_e = \phi_b$ and puts another geometric constraint on the amplitudes,

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} \quad \rightarrow \quad \boldsymbol{k} \times \boldsymbol{b} = (\omega / c^2) \boldsymbol{e} \quad \rightarrow \quad \hat{\boldsymbol{k}} \times (c\, \boldsymbol{b}) = -\boldsymbol{e}. \tag{24}$$

It is straightforward to show that the above geometric constraint is equivalent to the previous one,

$$\hat{\boldsymbol{k}} \times (c\,\boldsymbol{b}) = \hat{\boldsymbol{k}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{e}) = \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{e}) - (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}})\boldsymbol{e} = -\boldsymbol{e}. \tag{25}$$

Note that the above constraints imply that $\boldsymbol{k}$, $\boldsymbol{e}$, $\boldsymbol{b}$ are mutually orthogonal. In fact, $\boldsymbol{k}$, $\boldsymbol{e}$, $\boldsymbol{b}$ (the order is important here) form a right-hand orthogonal basis. With these constraints, the plane-wave solution can be rewritten as

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{e}\ \sin\left(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \phi_e\right), \tag{26}$$

$$\boldsymbol{B}(x, y, z, t) = \frac{1}{c}\,\hat{\boldsymbol{k}} \times \boldsymbol{e}\ \sin\left(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \phi_e\right) = \frac{1}{c}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}(x, y, z, t). \tag{27}$$

It shall be clear that both fields are closely related to each other and the light should be treated as the electromagnetic wave. Because the magnetic field can be dictated by the electric field, one only needs to count the independent modes of the electric field. In three dimensions, the transverse directions with respect to a given wave number $\boldsymbol{k}$ is two and the amplitude $\boldsymbol{e}$ allows two independent solutions. This explains why light has two polarizations – secretly embedded in the Maxwell equations already.