14.

X : 救護車到達的位置 $\sim Unif(0,L)$

Y : 事故發生的位置 $\sim Unif(0,L)$

$Let\ \ \mathrm{D} = |X - Y|$

$$F_D(d) = P(D < d) = P(|X - Y| < d)$$
$$= P(Y < X < Y + d) + P(X < Y < X + d)$$
$$= \frac{2}{L^2} \int_0^L \int_y^{\min(y+d,L)} dxdy$$
$$= \frac{2}{L^2} \left[\int_0^{L-d} \int_y^{y+d} dxdy + \int_{L-d}^{L} \int_y^{L} dxdy \right]$$
$$= \frac{d}{L}\left(2 - \frac{d}{L}\right) \qquad 0 < d < L$$

38.

		X				
		1	2	3	4	5
Y	1	P(X=1,Y=1)= $\frac{1}{5}$	P(2,1)= $\frac{1}{5}\cdot\frac{1}{2}$	P(3,1)= $\frac{1}{5}\cdot\frac{1}{3}$	P(4,1)= $\frac{1}{5}\cdot\frac{1}{4}$	P(5,1)= $\frac{1}{5}\cdot\frac{1}{5}$
	2		P(2,2)= $\frac{1}{5}\cdot\frac{1}{2}$	P(3,2)= $\frac{1}{5}\cdot\frac{1}{3}$	P(4,2)= $\frac{1}{5}\cdot\frac{1}{4}$	P(5,2)= $\frac{1}{5}\cdot\frac{1}{5}$
	3			P(3,3)= $\frac{1}{5}\cdot\frac{1}{3}$	P(4,3)= $\frac{1}{5}\cdot\frac{1}{4}$	P(5,3)= $\frac{1}{5}\cdot\frac{1}{5}$
	4				P(4,4)= $\frac{1}{5}\cdot\frac{1}{4}$	P(5,4)= $\frac{1}{5}\cdot\frac{1}{5}$
	5					P(5,5)= $\frac{1}{5}\cdot\frac{1}{5}$

(a)

$$P(X = j, Y = i) = \frac{1}{5}\cdot\frac{1}{j} \qquad j = 1,\cdots,5 \qquad i = 1,\cdots,j$$

(b)

$$P(X = j \mid Y = i) = \frac{P(X = j, Y = i)}{P(Y = i)} = \frac{\frac{1}{5}\cdot\frac{1}{j}}{\sum_{\mathrm{k=i}}^{5} \frac{1}{5k}} = \frac{\frac{1}{j}}{\sum_{\mathrm{k=i}}^{5} \frac{1}{k}} \qquad 5 \ge j \ge i$$

(c)

$P(X=5,Y=5)=\frac{1}{25} \qquad P(X=5)=\frac{1}{5} \qquad P(Y=5)=\frac{1}{25}$

$\Rightarrow P(X=5,Y=5)\neq P(X=5)P(Y=5)$

Hence, X and Y are not independent.

41.

(a)

$$f_{X|Y}(x\mid y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dx}=(y+1)^2xe^{-x(y+1)} \qquad x>0$$

$$f_{Y|X}(y\mid x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dy}=xe^{-xy} \qquad y>0$$

(b)

$$F_Z(z)=P(Z<z)=P(XY<z)$$

$$=\int_0^\infty\int_0^{\frac{z}{x}}xe^{-x(y+1)}\,dydx=\int_0^\infty(1-e^{-z})\cdot e^{-x}dx=1-e^{-z}$$

$$\Rightarrow f_Z(z)=e^{-z} \qquad z>0$$

58.

$X_1\sim Exp(\lambda)$

$X_2\sim Exp(\lambda)$

$$f_{X_1,X_2}(x_1,x_2)=\lambda^2\cdot e^{-\lambda\cdot(x_1+x_2)} \qquad x_1>0 \quad x_2>0$$

$Y_1=X_1+X_2$

$Y_2=e^{X_1}$

$\Rightarrow X_1=\ln Y_2 \qquad X_2=Y_1-\ln Y_2$

$$\Rightarrow |J|=\left|\begin{vmatrix}0 & \frac{1}{y_2}\\ 1 & -\frac{1}{y_2}\end{vmatrix}\right|=\frac{1}{y_2}$$

$$h_{Y_1,Y_2}(y_1,y_2)=f(\ln y_2,y_1-\ln y_2)\cdot|J|=\frac{\lambda^2}{y_2}e^{-\lambda\cdot y_1} \qquad y_1>\ln y_2 \quad y_2>1$$

Theoretical Exercises

14.

$X \sim Geo(p)$

$Y \sim Geo(p)$

(a)

給定 $X+Y=n$ 下，表第2次成功發生在第n次試行，則前$n-1$次試行中只需1次成功發生，故$P(X=i \mid X+Y=n)=\dfrac{1}{C_1^{n-1}}=\dfrac{1}{n-1}$

(b)

$$P(X=i \mid X+Y=n)=\frac{P(X=i,Y=n-i)}{P(X+Y=n)}=\frac{p(1-p)^{i-1}\cdot p(1-p)^{n-i-1}}{C_1^{n-1}\cdot p^2(1-p)^{n-2}}=\frac{1}{n-1}$$

22.

$W \sim \Gamma(t,\beta)$

$$f_{W|X_1=x_1,\cdots,X_n=x_n}(w \mid X_1=x_1,\cdots,X_n=x_n)$$

$$=\frac{f(w,x_1,\cdots,x_n)}{f(x_1,\cdots,x_n)}=\frac{f(x_1,\cdots,x_n \mid W=w)\cdot f(w)}{f(x_1,\cdots,x_n)}$$

$$=\left(\prod_{i=1}^{n} w\cdot e^{-wx_i}\right)\cdot\frac{e^{-\beta w}\cdot\beta^t\cdot w^{t-1}}{\Gamma(t)}\cdot\frac{1}{f(x_1,\cdots,x_n)}$$

$$=\underbrace{\frac{\beta^t}{\Gamma(t)\cdot f(x_1,\cdots,x_n)}}_{\text{與}w\text{無關}}\cdot\underbrace{w^{t+n-1}\cdot e^{-(\beta+\sum_{i=1}^{n}x_i)w}}_{(1)}$$

由(1)可觀察到

$$W \mid X_1=x_1,\cdots,X_n=x_n \sim \Gamma(t+n,\beta+\sum_{i=1}^{n}x_i)$$

31.

$By\ Lecture\ notes\ p.6-43$

$X_1,\cdots,X_n \sim Unif(0,1)$

$$f_{X_{(k-1)},X_{(k)}}(s,w)=\frac{n!}{(k-2)!(n-k)!}s^{k-2}(1-w)^{n-k}$$

$For\ 1\le k\le n+1 \qquad X_{(0)}\equiv 0\,,\,X_{(n+1)}\equiv 1$

$Let\ U = X_{(k)}-X_{(k-1)}=W-S$

$\Rightarrow W=U+S \quad and \quad 0<s<s+u<1$

$$f_U(u)=\int f_{X_{(k-1)},X_{(k)}}(s,u+s)\,ds$$

$$=\int_0^{1-u}\frac{n!}{(k-2)!(n-k)!}s^{k-2}(1-u-s)^{n-k}ds \qquad [Let\ \ s=(1-u)x \Rightarrow ds=(1-u)dx]$$

$$=\int_0^1\frac{n!}{(k-2)!(n-k)!}(1-u)^{k-2}x^{k-2}(1-u)^{n-k}(1-x)^{n-k}(1-u)dx$$

$$=\frac{n!}{(k-2)!(n-k)!}\cdot(1-u)^{n-1}\cdot\int_0^1 x^{k-2}(1-x)^{n-k}dx$$

$$=\frac{n!}{(k-2)!(n-k)!}\cdot(1-u)^{n-1}\cdot\frac{(k-2)!(n-k)!}{(n-1)!} \qquad \left[\int_0^1 x^{k-2}(1-x)^{n-k}dx=Beta(k-1,n-k+1)\right]$$

$$=n\cdot(1-u)^{n-1} \quad 0<u<1 \qquad \left(\ In\ fact, U\sim Beta(1,n)\ \right)$$

$$\Rightarrow P(X_{(k)}-X_{(k-1)}>t)=P(U>t)=\int_t^1 n\cdot(1-u)^{n-1}du=(1-t)^n$$

34.

$Let\ Y_1=X_{(1)} \quad Y_n=X_{(n)}$

$$f_{Y_1,Y_n}(y_1,y_n)=n(n-1)\cdot(y_n-y_1)^{n-2}$$

$$M=\frac{Y_1+Y_n}{2} \qquad R=Y_n-Y_1$$

$$\Rightarrow\ Y_1=\frac{2M-R}{2} \qquad Y_n=\frac{2M+R}{2} \qquad and \quad |J|=1$$

$$h_{R,M}(r,m)=f\left(\frac{2m-r}{2},\frac{2m+r}{2}\right)\cdot|J|=n\cdot(n-1)\cdot r^{n-2} \qquad 0<r<1 \qquad \frac{r}{2}<m<1-\frac{r}{2}$$