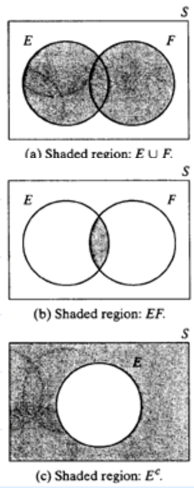
Sample Space and Events

- *Sample Space* Ω: the set of all possible outcomes in a random phenomenon. Examples:
  1. Sex of a newborn child: Ω = {girl, boy}
  2. The order of finish in a race among the 7 horses 1, 2, …, 7:

     Ω = {all 7! Permutations of (1, 2, 3, 4, 5, 6, 7)}
  3. Flipping two coins: Ω = {(H, H), (H, T), (T, H), (T, T)}
  4. Lifetime of a transistor: $\Omega = [0, \infty)$
- *Event*: Any (measurable) subset of Ω is an event. Examples:
  1. A={girl}: the event - child is a girl.
  2. A={all outcomes in Ω starting with a 3}: the event - horse 3 wins the race.
  3. A={(H, H), (H, T)}: the event - head appears on the 1st coin.
  4. A=[0, 5]: the event - transistor does not last longer than 5 hours.
  - an event occurs: outcome $\in$ the event
  - Q: How many different events if $\#\Omega = n < \infty$?

- Set Operations of Events
  - Union. $C = A \cup B \Rightarrow$ C: either A or B occurs
  - Intersection. $C = A \cap B \Rightarrow$ C: both A and B occur
  - Complement. $C = A^c \Rightarrow$ C: A does not occur
  - Mutually Exclusive. $A \cap B = \emptyset \Rightarrow$ A and B have no outcomes in common.
  - Definitions of union and intersection for more than two events can be defined in a similar manner

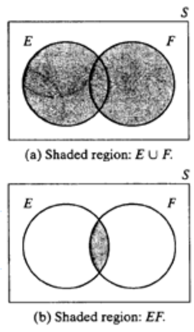


- Some Simple Rules of Set Operations
  - Commutative Laws. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
  - Associative Laws. $(A \cup B) \cup C = A \cup (B \cup C)$

    $(A \cap B) \cap C = A \cap (B \cap C)$.
  - Distributive Laws. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

    $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  - DeMorgan's Laws.

    $$(\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c \quad \text{and} \quad (\cap_{i=1}^n A_i)^c = \cup_{i=1}^n A_i^c.$$



❖ **Reading**: textbook, Sec 2.2

Probability Measure

- The Classical Approach

  Sample Space Ω is a *finite* set

  Probability: For an event A,

$$P(A) = \frac{\#A}{\#\Omega}$$

➢ Example (Roulette):

- Ω = {0, 00, 1, 2, 3, 4, …, 35, 36}
- P({Red Outcome}) = 18/38 = 9/19.

➢ Example (Birthday Problem): n people gather at a party. What is the probability that they all have different birthdays?

- Ω = lists of n from {1, 2, 3, …, 365}
- A = {all permutations}
- $P_n(A) = (365)_n/365^n$
-

n	8	16	24	32	40
$P_n(A)$	.926	.716	.462	.247	.109

➢ Inadequacy of the Classical Approach

$$P(A) = \frac{\#A}{\#\Omega}$$

- It requires:
  - Finite Ω
  - Symmetric Outcomes
- Example (Sampling Proportional to Size):
  - N invoices.
  - Sample $n < N$.
  - Pick large ones with higher probability.
  - Note: Finite Ω, but *non equally-likely* outcomes.
- Example (Waiting for a success):
  - Play roulette until a win.
  - $\Omega = \{1, 2, 3, \ldots\}$.
  - P = ??
- Example (Uniform Spinner):
  - Random Angle (in radians).
  - $\Omega = (-\pi, \pi]$.
  - P = ??

- The Modern Approach
  - A *probability measure* on Ω is a function P from subsets of Ω to the real number that satisfies the following axioms:

    (Ax1) Non-negativity. For any event A, $P(A)\geq 0$.

    (Ax2) Total one. $P(\Omega)=1$

    (Ax3) Additivity. If A_1, A_2, …, is a sequence of mutually exclusive events, i.e., $A_i \cap A_j = \emptyset$ when $i\neq j$, then
    $$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$
    - Notes:
      - These axioms restrict probabilities, but do not define them.
      - Probability is a property of events.
  - Define Probability Measures in a Discrete Sample Space.
    - **Q**: Is it required to define probabilities on every events? (e.g., n possible outcomes in Ω, 2^n-1 possible events)
    - Suppose $\Omega = \{\omega_1, \omega_2, \ldots\}$, *finite* or *countably infinite*, let $p: \Omega \to R$ satisfy
    $$p(w) \geq 0 \text{ for all } \omega \in \Omega \quad \text{and} \quad \textstyle\sum_{\omega\in\Omega} p(w) = 1.$$

Let
$$P(A) = \textstyle\sum_{\omega\in A} p(\omega)$$
for $A\subset\Omega$, then P is a probability measure. (exercise)

(**Q**: how to define p?)

- Example: In the classical approach, $p(\omega) = 1/\#\Omega$. For example, throw a fair dice, $\Omega=\{1, \ldots, 6\}$, $p(1)=\ldots=p(6)=1/6$ and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)=3/6=1/2$.
- Example (non equally-likely events): Throwing an unfair dice might have $p(1)=3/8$, $p(2)=p(3)=\ldots=p(6)=1/8$, and $P(\text{odd})=P(\{1, 3, 5\})=p(1)+p(3)+p(5)= 5/8$. (c.f., Example in LNp.2-4)
- Example (Waiting for Success – Play Roulette Until a Win):
  - Let $r=9/19$ and $q=1-r=10/19$
  - $\Omega = \{1, 2, 3, \ldots\}$
  - Intuitively, $p(1)=r$, $p(2)=qr$, $p(3)=q^2r$, …, $p(n)=q^{n-1}r$, … >0, and
  $$\sum_{n=1}^{\infty} p(n) = \sum_{n=1}^{\infty} rq^{n-1} = \frac{r}{1-q} = 1.$$

- For an event $A \subset \Omega$, let

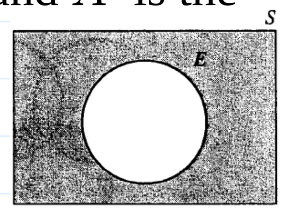
$$P(A) = \sum_{n \in A} p(n).$$

For example, Odd={1, 3, 5, 7, …}

$$P(\text{Odd}) = \sum_{k=0}^{\infty} p(2k+1) = \sum_{k=0}^{\infty} rq^{(2k+1)-1} = r\sum_{k=0}^{\infty} q^{2k} = r/(1-q^2) = 19/29.$$

- Some Consequences of the 3 Axioms
  - Proposition: If A is an event in a sample space Ω and A^c is the complement of A, then

$$P(A^c) = 1 - P(A).$$



  - Proposition: For any sample space Ω, the probability of the empty set is zero, i.e.,

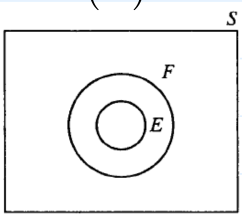
$$P(\emptyset) = 0.$$

  - Proportion: For any finite sequence of mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

  - Proposition: If A and B are events in a sample space Ω and $A \subset B$, then

$$P(A) \le P(B) \quad \text{and} \quad P(B - A) = P(B \cap A^c) = P(B) - P(A).$$

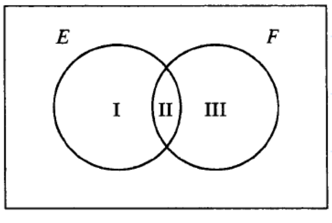


  - Proposition: If A is an event in a sample space Ω, then

$$0 \le P(A) \le 1.$$

  - Proposition: If A and B are two events in a sample space Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



- Proposition: If $A_1, A_2, \ldots, A_n$ are events in a sample space Ω, then

$$P(A_1 \cup \cdots \cup A_n) \leq P(A_1) + \cdots + P(A_n).$$

- Proposition (inclusion-exclusion identity): If $A_1, A_2, \ldots, A_n$ are any n events, let

$$\begin{aligned}
\sigma_1 &= \sum_{i=1}^{n} P(A_i), \\
\sigma_2 &= \sum_{1\leq i<j\leq n} P(A_i \cap A_j), \\
\sigma_3 &= \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k), \\
\cdots &= \cdots \\
\sigma_k &= \sum_{1\leq i_1<\cdots<i_k\leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}) \\
\cdots &= \cdots \\
\sigma_n &= P(A_1 \cap A_2 \cap \cdots \cap A_n).
\end{aligned}$$

Q: For an outcome w contained in m out of the n events, how many times is its probability repetitively counted in $\sigma_1, \ldots, \sigma_n$?

then,

$$P(A_1 \cup \cdots \cup A_n) = \sigma_1 - \sigma_2 + \sigma_3 - \cdots + (-1)^{k+1}\sigma_k + \cdots + (-1)^{n+1}\sigma_n.$$

- Notes:
  - There are $\binom{n}{k}$ summands in σ_k
  - In symmetric examples,

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k)$$

  - It can be shown that

$$\begin{aligned}
P(A_1 \cup \cdots \cup A_n) &\leq \sigma_1 \\
P(A_1 \cup \cdots \cup A_n) &\geq \sigma_1 - \sigma_2 \\
P(A_1 \cup \cdots \cup A_n) &\leq \sigma_1 - \sigma_2 + \sigma_3 \\
\cdots \quad & \cdots \quad \cdots
\end{aligned}$$

- Example (The Matching Problem).
  - Applications: (a) Taste Testing. (b) Gift Exchange.
  - Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of 1, 2, …, n. Thus, $\#\Omega = n!$.
  - Let

$$A_j = \{\omega: i_j = j\} \quad \text{and} \quad A = \cup_{i=1}^{n} A_i,$$

Q: $P(A)$=? (What would you expect when n is large?)

  - By symmetry,

$$\sigma_k = \binom{n}{k} P(A_1 \cap \cdots \cap A_k),$$

  - Here,

$$\begin{aligned} P(A_1) &= \frac{1 \times (n-1)!}{n!} = \frac{1}{n}, \\ P(A_1 \cap A_2) &= \frac{(n-2)!}{n!} = \frac{1}{(n)_2}, \\ \cdots &= \cdots, \\ P(A_1 \cap \cdots \cap A_k) &= \frac{(n-k)!}{n!} = \frac{1}{(n)_k}. \end{aligned}$$

for $k = 1, \ldots, n$.

  - So, $\sigma_k = \binom{n}{k} \frac{1}{(n)_k} = \frac{1}{k!},$

$$P(A) = \sigma_1 - \sigma_2 + \cdots + (-1)^{n+1}\sigma_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!},$$

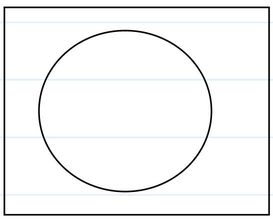
$$P(A) = 1 - \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx 1 - \frac{1}{e} \quad \Rightarrow P(A^c) \approx e^{-1}$$

  - Note: approximation accurate to 3 decimal places if $n \geq 6$.

➤ Proportion: If $A_1, A_2, \ldots,$ is a *partition* of Ω, i.e.,

1. $\cup_{i=1}^{\infty} A_i = \Omega,$
2. $A_1, A_2, \ldots,$ are mutually exclusive,

then, for any event $A \subset \Omega$,



$$P(A) = \sum_{i=1}^{\infty} P(A \cap A_i).$$

- Monotone Sequences

➢ **Q**: How to define probability in a continuous sample space?

➢Definition: A sequence of events A_1, A_2, …, is called *increasing* if

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$$

and *decreasing* if

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$$

The limit of an increasing sequence is defined as

$$\lim_{n\to\infty} A_n = \cup_{i=1}^{\infty} A_i$$

and the limit of an decreasing sequence is

$$\lim_{n\to\infty} A_n = \cap_{i=1}^{\infty} A_i$$

➢Example: If Ω=R and $A_k = (-\infty, 1/k)$, then A_k's are decreasing and

$$\lim_{k\to\infty} A_k = \{\omega : \omega < 1/k \text{ for all } k \in \mathcal{Z}_+\} = (-\infty, 0].$$

➢Proportion: If A_1, A_2, …, is increasing or decreasing, then

$$\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} A_n^c$$

➢Proportion: If A_1, A_2, …, is increasing or decreasing, then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right).$$

➢Example (Uniform Spinner): Let Ω = (–π, π]. Define

$$P((a,b]) = \frac{b-a}{2\pi}.$$

for subintervals $(a, b]$⊂Ω. Then, extend P to other subsets using the 3 axioms. For example, if $-\pi < a < b < \pi$,

$$P([a,b]) = P\left(\left(\bigcap_{k=1}^{\infty}(a-\frac{1}{k},b]\right)\cap\Omega\right) = P\left(\bigcap_{k=1}^{\infty}\left((a-\frac{1}{k},b]\cap\Omega\right)\right)$$
$$= \lim_{k\to\infty} P\left((a-\frac{1}{k},b]\cap\Omega\right)$$
$$= \lim_{k\to\infty}\frac{1}{2\pi}(b-a+\frac{1}{k}) = \frac{b-a}{2\pi}.$$

- Some notes
  - $P(\{a\}) = P([a,b]-(a,b]) = P([a,b]) - P((a,b]) = 0.$
  - If $C=\{\omega_1, \omega_2, \ldots\}\subset\Omega$, then
  $$P(C) = \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0+0+\cdots = 0.$$
  - The probability of a rational outcome is zero

• Objective vs. Subjective Interpretations of Probability

➢ **Q**: What do we mean if we say that the probability of rain tomorrow is 40%?

Objective: Long run relative frequencies
Subjective: Chosen to reflect opinion

➢ The Objective (Frequency) Interpretation

- Through Experiment: Imagine the experiment repeated N times. For an event A, let

$$N_A = \#\text{ occurrences of } A.$$

Then,
$$P(A) \equiv \lim_{N\to\infty}\frac{N_A}{N}.$$

- Example (Coin Tossing):

N	100	1000	10000	100000
N_H	55	493	5143	50329
N_H/N	.550	.493	.514	.503

The result is consistent with $P(H)$=0.5.

➢ The Subjective Interpretation

- Strategy: Assess probabilities by imagining bets
- Examples:
  - Peter is willing to give two to one odds that it will rain tomorrow. His subjective probability for rain tomorrow is at least 2/3

- Paul accepts the bet. His subjective probability for rain tomorrow is at most 1/3

- Probabilities are simply personal measures of how likely we think it is that a certain event will occur
- This can be applied even when the idea of repeated experiments is not feasible

❖ **Reading**: textbook, Sec. 2.3~2.7