Chapter 3  Rigid Bodies : Equivalent System of Forces

3.1 Introduction

- principle of transmissibility
- moment of a force about a point
- moment of a force about an axis
- body : combination of large number of particles
- rigid body : does not deform under force acting on the body

3.2 External and Internal Forces

forces acting on the body may be separated into two groups
1. external forces : represent the action of other bodies on the rigid body
2. internal forces : the forces which hold together the particles forming the rigid body

3.3 Principle of Transmissibility, Equivalent Forces

the condition of equilibrium or of motion of a rigid body will remain unchanged if a force \( F \) acting at a given point of the rigid body is replaced by a force \( F' \) of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action

the two forces \( F \) and \( F' \) have the same effect on the rigid body and are said to be equivalent
the principle of transmissibility and the concept of equivalent force have limitations

![Diagram](image)

from the point of view of mechanics of rigid bodies, the system shown above are thus equivalent, but the internal forces and deformations are clearly different: (a) tension, (b) and (c) no stress

similarly for compression

thus, while the principle of transmissibility may be used freely to determine the conditions of motion or equilibrium of rigid bodies and the compute the external forces action on these bodies, it should be avoided, or at least used with care, in determining internal force and deformation

3.4 Vector Product of Two Vectors

\[ V = P \times Q \]

1. \( V \perp P, \quad V \perp Q \)
2. \( V = PQ \sin \theta \)
3. the direction of \( V \) obtained from the right hand rule

vector product also known as cross product

i. vector product are not commutative

\[ P \times Q = -Q \times P \]
ii. distributive property can be applied

\[ P \times (Q_1 + Q_2) = P \times Q_1 + P \times Q_2 \]

iii. associative property does not applied to vector product

\[ (P \times Q) \times S \neq P \times (Q \times S) \]

3.5 Vector Products Expressed in Terms of Rectangular Components

\[ i \times i = 0 \]
\[ i \times j = k \]
\[ i \times k = -j \quad \text{etc.} \]
\[ V = P \times Q = (P_x i + P_y j + P_z k) \times (Q_x i + Q_y j + Q_z k) \]
\[ V_x i + V_y j + V_z k = (P_y Q_z - P_z Q_y)i + (P_z Q_x - P_x Q_z)j \]
\[ + (P_x Q_y - P_y Q_x)k \]

i.e.
\[ V_x = P_y Q_z - P_z Q_y \quad V_y = P_z Q_x - P_x Q_z \]
\[ V_z = P_x Q_y - P_y Q_x \]

or
\[ V = P \times Q = \begin{vmatrix} i & j & k \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \]

3.6 Moment of a Force about a Point

defined the moment of \( F \)

about \( O \) as the vector product

of \( r \) and \( F \)

\[ M_0 = r \times F \quad r = OA \]
\[ M_0 = r F \sin \theta = F d \]

where \( d \) is the perpendicular distance from \( O \) to the line of action of the force \( F \)
3.7 Varignon’s Theorem

the moment about a given point $O$ of a resultant of several concurrent forces is equal to the sum of the moments of various forces about the same point $O$

$$F = F_1 + F_2 + F_3 + \cdots$$

$$M_0 = r \times F = r \times (F_1 + F_2 + F_3 + \cdots)$$

$$= r \times F_1 + r \times F_2 + r \times F_3 + \cdots$$

3.8 Rectangular Components of the moment of a Force

$$r = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

$$F = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$$

$$M_0 = r \times F = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$M_0 = M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k}$$

i.e. $M_x = y F_z - z F_y \quad M_y = z F_x - x F_z \quad M_z = x F_y - y F_x$

if we need to compute the moment $M_B$ about an arbitrary at point $B$ of a force $F$ applied at $A$

$$M_B = BA \times F$$
\[ BA = \Delta x \, i + \Delta y \, j + \Delta z \, k \]
\[ x_{A/B} = x_A - x_B \quad y_{A/B} = y_A - y_B \]
\[ z_{A/B} = z_A - z_B \]
\[ BA = r_A - r_B = \Delta r \]

\[ \therefore \quad M_B = \Delta r \times F = \begin{vmatrix} i & j & k \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \]

for two dimensional case, \( F_z = 0, \, z = 0 \)

\[ M_0 = (x \, F_y - y \, F_x) \, k \]
\[ M_0 = x \, F_y - y \, F_x = M_z \]
\[ M_x = M_y = 0 \]

or \[ M_B = [(x_A - x_B) \, F_y - (y_A - y_B) \, F_x] \, k \]
\[ M_B = M_z = (x_A - x_B) \, F_y - (y_A - y_B) \, F_x \]
\[ M_x = M_y = 0 \]

Sample Problem 3.1

determine \( M_0 \)

(a) \( M_0 = F \, d = 500 \times 600 \cos 60^\circ \)
\[ = 500 \times 300 = 150,000 \, \text{N-mm} \]
\[ = 150 \, \text{N-m} \]

(b) \( M_0 = F \, d = F \times 600 \sin 60^\circ \)
\[ 150 = F \times 0.5196 \]
\[ F = 288.68 \, \text{N} \]
(c) \( M_0 = Fd \quad \text{and} \quad F_{\min} \rightarrow d_{\max} \)

i.e. \( F \perp OA \quad d_{\max} = OA = 0.6 \text{ m} \)

\[ 150 = F_{\min} \times 0.6 \]

\[ F_{\min} = 250 \text{ N} \quad \triangleleft \quad 30^\circ \]

(d) \( M_0 = Fd \)

\[ 150 = 1200d \quad d = 0.125 \text{ m} \]

\[ OB = d / \cos 60^\circ = 0.25 \text{ m} = 250 \text{ mm} \]

(e) none of (b), (c), (d) is equivalent, although \( M_0 \) is equal in each case, but \( F \) is not equal

Sample Problem 3.2

determine \( M_B \)

\[ M_B = r_{A/B} \times F \]

\[ r_{A/B} = BA = -0.2i + 0.16j \text{ (m)} \]

\[ F = 800 \cos 60^\circ \, i + 800 \sin 60^\circ \, j \]

\[ = 400\, i + 693\, j \text{ (N)} \]

\[ M_B = (-0.2i + 0.16j) \times (400\, i + 693\, j) \]

\[ = -138.6\, k - 64\, k = -202.6\, k \text{ (N-m)} \]

\[ = 202.6 \text{ N-m (}\triangleleft\text{)} \]

Sample Problem 3.3

\[ P = 40 \text{ N} \quad \alpha = 25^\circ \]

\[ P_x = 40 \cos 25^\circ = 36.252 \text{ N} \]

\[ P_y = 40 \sin 25^\circ = 16.905 \text{ N} \]

\[ \therefore M_0 = -x \, P_y - y \, P_x \]

\[ = -0.2 \times 16.905 - 0.6 \times 36.252 = -25.1 \text{ N-m} \]

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Sample Problem 3.4

\( T_{CD} = 200 \text{ N} \)

find \( M_A \) of \( T_{CD} \) at \( C \)

\[ M_A = r_{CA} \times T_{CD} \]

\( A(0, 0, 0.32) \quad C(0.3, 0, 0.4) \quad D(0, 0.24, 0.08) \) unit : m

\[ r_{CA} = AC = 0.3\, i + 0.08\, k \ (\text{m}) \]

\[ CD = -0.3\, i + 0.24\, j - 0.32\, k \ (\text{m}) \]

\[ CD = (0.3^2 + 0.24^2 + 0.32^2)^{\frac{1}{2}} = 0.5 \text{ m} \]

\[ \lambda_{CD} = CD / CD = -0.6\, i + 0.48\, j - 0.64\, k \]

\[ T_{CD} = T_{CD} \lambda_{CD} = 200 \cdot (-0.6\, i + 0.48\, j - 0.64\, k) \]

\[ = -120\, i + 96\, j - 128\, k \ (\text{N}) \]

\[ M_A = \begin{vmatrix} i & j & k \end{vmatrix} \]

\[ = \begin{vmatrix} 0.3 & 0 & 0.08 \end{vmatrix} = -7.68\, i + 28.8\, j + 28.8\, k \ (\text{N-m}) \]

\[ \begin{vmatrix} -120 & 96 & -128 \end{vmatrix} \]

3.9 Scalar Product of Two Vector (dot product)

\[ P \cdot Q = P Q \cos \theta \ (\text{scalar}) \]

commutative property \( P \cdot Q = Q \cdot P \)

distributive property \( P \cdot (Q_1 + Q_2) = P \cdot Q_1 + P \cdot Q_2 \)

associative property cannot be applied, \( \cdot \ (P \cdot Q) \cdot S \) has no meaning

dot product of unit vectors \( (= 0 \ or \ 1) \)

\[ i \cdot i = 1 \times 1 \cos 0^\circ = 1 \quad j \cdot j = 1 \quad k \cdot k = 1 \]

\[ i \cdot i = j \cdot i = i \cdot k = k \cdot i = j \cdot k = k \cdot j = 0 \]

\[ P \cdot Q = (P_x i + P_y j + P_z k) \cdot (Q_x i + Q_y j + Q_z k) \]
\[
= P_x Q_x + P_y Q_y + P_z Q_z
\]
if \( P = Q \) \( P \cdot P = P_x^2 + P_y^2 + P_z^2 = P^2 \)

applications:
1. angle formed by two given vectors
\[
P = P_x i + P_y j + P_z k
\]
\[
Q = Q_x i + Q_y j + Q_z k
\]
\[
P \cdot Q = P Q \cos \theta = P_x Q_x + P_y Q_y + P_z Q_z
\]
\[
\cos \theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{P \cdot Q}
\]
where \( P = (P_x^2 + P_y^2 + P_z^2)^{\frac{1}{2}} \)
\( Q = (Q_x^2 + Q_y^2 + Q_z^2)^{\frac{1}{2}} \)
2. projection of a vector on a given axis
\[
P_{OL} = OA = P \cos \theta
\]
\[
P \cdot Q = P Q \cos \theta = P_{OL} Q
\]
\[
P_{OL} = \frac{P \cdot Q}{Q} = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{Q}
\]
now, if we choose \( Q \) is an unit vector
\[
Q = \lambda_{OL} \quad Q = 1
\]
\[
P_{OL} = P \cdot \lambda_{OL}
\]
\[
\lambda_{OL} = \cos \theta_x i + \cos \theta_y j + \cos \theta_z k
\]
\[
\therefore \quad P_{OL} = P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z
\]

3.10 Mixed Triple Product of Three Vectors
\[
S \cdot (P \times Q) \quad \text{scalar triple product}
\]
\[
S \times (P \times Q) \quad \text{vector triple product}
\]
(used in dynamics)
\[ U = S \cdot (P \times Q) = S \cdot V \]

\( U \) = the volume of the parallelogram

having the vector \( S, P, Q \) for sides

if \( S, P, Q \) are coplanar, the scalar triple product will be zero

\[
S \cdot (P \times Q) = P \cdot (Q \times S) = Q \cdot (S \times P) \\
= -S \cdot (Q \times P) = -Q \cdot (P \times S) = -P \cdot (S \times Q)
\]

\[
S \cdot (P \times Q) = S \cdot V = S_x V_x + S_y V_y + S_z V_z \\
= S_x (P_y Q_z - P_z Q_y) + S_y (P_z Q_x - P_x Q_z) + S_z (P_x Q_y - P_y Q_x)
\]

\[
S \cdot (P \times Q) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}
\]

3.11 Moment of a Force about a Given Axis

\[ M_0 = r \times F \]

define the moment \( M_{OL} \) of \( F \) about \( OL \) as the projection \( OC \) of the moment \( M_0 \) on the axis \( OL \)

\[ M_{OL} = \lambda \cdot M_0 = \lambda \cdot (r \times F) \]

\[
\begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}
\]

resolve \( F \) into \( F_1 \) and \( F_2 \) in which \( F_1 \parallel OL \) and \( F_2 \) on the plane \( \perp OL \)

\[ F = F_1 + F_2 \]
and \( \mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 \) where \( \mathbf{r}_1 \parallel \lambda \)

\[
\mathbf{M}_{OL} = \lambda \cdot (\mathbf{r} \times \mathbf{F}) = \lambda \cdot [(\mathbf{r}_1 + \mathbf{r}_2) \times (\mathbf{F}_1 + \mathbf{F}_2)]
\]

\[
= \lambda \cdot (\mathbf{r}_1 \times \mathbf{F}_1) + \lambda \cdot (\mathbf{r}_1 \times \mathbf{F}_2) + \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_1) + \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_2)
\]

\[
= \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_2) = |\mathbf{r}_2 \times \mathbf{F}_2|
\]

the moment \( \mathbf{M}_{OL} \) of \( \mathbf{F} \) about \( \mathbf{OL} \) measures the tendency of the force \( \mathbf{F} \) to impart to the rigid body a motion or rotation about the fixed axis \( \mathbf{OL} \)

\[
\therefore \quad \mathbf{M} \cdot \mathbf{i} = M_x = yF_z - zF_y
\]

\[
\mathbf{M} \cdot \mathbf{j} = M_y = zF_x - xF_z \quad \text{moment about the coord. axes}
\]

\[
\mathbf{M} \cdot \mathbf{k} = M_z = xF_y - yF_x
\]

the moment components \( M_x, M_y \) and \( M_z \) of \( \mathbf{F} \) about the coordinate axes measure the tendency of \( \mathbf{F} \) to impart to the rigid body a motion of rotation about the \( x, y \) and \( z \) axes, respectively.

the moment of \( \mathbf{F} \) applied at \( \mathbf{A} \) about an axis which does not pass through \( \mathbf{O} \) is obtained by choosing an arbitrary point \( \mathbf{B} \) on the axis

\[
\mathbf{M}_{BL} = \lambda \cdot \mathbf{M}_B = \lambda \cdot (\Delta \mathbf{r} \times \mathbf{F})
\]

\[
\Delta \mathbf{r} = \mathbf{r}_A - \mathbf{r}_B
\]

\[
\mathbf{M}_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \Delta x & \Delta y & \Delta z \\ F_x & F_y & F_z \end{vmatrix}
\]

\[
\Delta x = x_A - x_B \quad \Delta y = y_A - y_B \quad \Delta z = z_A - z_B
\]

the moment of \( \mathbf{F} \) about an axis is independent of the choice of the point on the axis
choose another point \( \mathbf{C} \)

\[
\mathbf{M}_{CL} = \lambda \cdot [(\mathbf{r}_A - \mathbf{r}_C) \times \mathbf{F}]
\]

\[
\mathbf{r}_A - \mathbf{r}_C = (\mathbf{r}_A - \mathbf{r}_B) + (\mathbf{r}_B - \mathbf{r}_C)
\]

\[
\mathbf{M}_{CL} = \lambda \cdot [(\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}] + \lambda \cdot [(\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{F}]
\]

\[
= \lambda \cdot (\Delta \mathbf{r} \times \mathbf{F}) = \mathbf{M}_{BL}
\]

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Sample Problem 3.5

a cube of side \( a \), determine

(a) \( M_A \)  
(b) \( M_{AB} \)

(c) \( M_{AG} \)  
(d) \( d \) from \( AG \) to \( FC \)

\[
P = P \lambda_{FC} \quad F(a, 0, a) \quad C(a, a, 0) \\
\lambda_{FC} = FC / FC = (j - k) / \sqrt{2}
\]

(a) \[
M_A = AF \times P = (aj - ak) \times P(j - k) / \sqrt{2} \\
= aP(i + j + k) / \sqrt{2}
\]

(b) \[
M_{AB} = \lambda_{AB} \cdot M_A \\
\lambda_{AB} = i \\
= i \cdot aP(i + j + k) / \sqrt{2} = aP / \sqrt{2}
\]

(c) \[
M_{AG} = \lambda_{AG} \cdot M_A \\
A(0, a, a) \quad G(a, 0, 0) \quad AG = a(i - j - k) \quad AG = a \sqrt{3} \\
\lambda_{AG} = (i - j - k) / \sqrt{3} \\
M_{AG} = [(i - j - k) / \sqrt{3}] \cdot [aP(i + j + k) / \sqrt{2}] = -aP / \sqrt{6}
\]
or \[
M_{AG} = \lambda_{AG} \cdot (AF \times P) \\
= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a & -a & 0 \\ 0 & P / \sqrt{6} & -P / \sqrt{6} \end{vmatrix} = -aP / \sqrt{6}
\]

(d) \[
\therefore P \cdot \lambda_{AG} = [P(j - k) / \sqrt{2}] \cdot [(i - j - k) / \sqrt{3}] = 0
\]

\[
\therefore P \perp \lambda_{AG}
\]

\[
\therefore M_{AG} = |Pd| = aP / \sqrt{6} = Pd \\
\text{thus } d = a / \sqrt{6}
\]

3.12 Moment of a Couple

two forces \( F \) and \(-F\) having the same 
magnitude, parallel line of action, opposite 
direction are said to form a couple
\[ M_0 = \mathbf{r}_A \times F + \mathbf{r}_B \times (-F) \]
\[ = (\mathbf{r}_A - \mathbf{r}_B) \times F = \mathbf{r} \times F \]

where \( \mathbf{r} \) is the vector jointing the points of application of the two forces, \( M_0 \) is called the moment of couple

\[ M_0 = r F \sin \theta = F d \]

\( \mathbf{r} \) is independent of the origin \( O \), \( \therefore \) the moment \( M \) of a couple is a free vector, i.e. which may be applied at any point

if \( F_1 d_1 = F_2 d_2 \)

then the moment of couple is said to be equal

### 2.13 Equivalent Couples

three couples have the same effect on the box

two system of forces are equivalent if we can transform one of them into other by means of one or several of the following operations:

1. replacing two forces acting on the same particle by their resultant
2. resolving a force into two components
3. canceling two equal and opposite forces acting on the same particle
4. attaching to the same particle two equal and opposite forces
5. moving a force along its line of action
two couples having the same moment \( M \) are equivalent

1. two couples contained
   in the same plane

2. two couples contained
   in the parallel planes

3.14 Addition of Couples

\[
M = r \times R = r \times (F_1 + F_2) = r \times F_1 + r \times F_2 = M_1 + M_2
\]

3.15 Couples may be Represented by Vectors

the vectors representing a couple is called a couple vector

3.16 Resolution of a Given Force into a Force at \( O \) and a Couple

any force acting on a rigid body may be moved to an arbitrary point \( O \), provided that a couple is added, of moment equal to the moment of \( F \) about \( O \)

i.e. \( M_0 = r \times F \quad M_0 = Fd \)
and the combination obtained is referred to as a force-couple system

if the force $F$ had been moved from $A$ to a different point $O'$, then

$$M_{O'} = r' \times F \quad \text{[from (a) to (c)]}$$

$$= (r + s) \times F = r \times F + s \times F$$

$$= M_0 + s \times F \quad \text{[from (b) to (c)]}$$

replacing a force-couple system into a single force

$$d = \frac{M_0}{F}$$

in this case, $F$ must be $\perp M_0$

Sample Problem 3.6

determine $M$ of the couples

$$DE = 0.23 j - 0.3 k$$

$M_1$ for the 100 N force

$$M_1 = DE \times (-100 i)$$

$$= (0.23 j - 0.3 k) \times (-100 i)$$

$$= 30 j + 23 k \quad \text{(N-m)}$$

$M_2$ for the 150 N force

$$M_2 = DC \times (-30 k)$$

$$= 0.46 j \times (-150 i)$$

$$= -69 i \quad \text{(N-m)}$$
\[ M = M_1 + M_2 = -69i + 30j + 23k \quad \text{(N-m)} \]

alternate solution, add +100 N and -100 N at A, then

\[ M_x = -150 \times 0.46 = -69 \text{ N-m} \]
\[ M_y = 100 \times 0.3 = 30 \text{ N-m} \]
\[ M_z = 100 \times 0.23 = 23 \text{ N-m} \]

then \[ M = -69i + 30j + 23k \quad \text{(N-m)} \]

Sample Problem 3.7

replace the force and couple by a single equivalent force
due to the force, the moment about O is

\[ M_0 = 400 \times d = 400 \times 0.3 \cos 60^\circ \]
\[ = 60 \text{ N-m} \quad (2) \]
due to the couple

\[ M_0 = 200 \times 0.12 = 24 \text{ N-m} \quad (2) \]
then the total moment at O is

\[ M_0 = 84 \text{ N-m} \quad (2) \]
the distance of the application of the equivalent force is

\[ d = M_0 / F = 84 / 400 = 0.21 \text{ m} \]
and \[ OC = 0.21 / \cos 60^\circ = 0.42 \text{ m} \]

alternative method, move the (-24 N-m) couple to B

\[ d = M / F = 24 / 400 = 0.06 \text{ m} \]
\[ BC = 0.06 / \cos 60^\circ = 0.12 \text{ m} \]
\[ OC = OB + BC = 0.3 + 0.12 = 0.42 \text{ m} \]
3.17 Reduction of a System of Force to One Force and One Couple

consider a system of forces \( F_1, F_2, F_3, \ldots \) acting at the rigid body at the points \( A_1, A_2, A_3, \ldots \), the forces can be removed to a given point \( O \)

then the equivalent force-couple system is defined

\[
R = \sum F \quad M_0^R = \sum M_0 = \sum (r \times F)
\]

\( M_0^R \) is called the moment resultant of the system at \( O \)
it can also to a force and a couple at another point \( O' \)

\[
R = \sum F \quad M_{0, R} = M_0^R + s \times R
\]

for each force, the position and force components is

\[
r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\]

\[
F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}
\]

and the resultant force and moment can be written as

\[
R = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}
\]

\[
M_{0, R} = M_x^R\mathbf{i} + M_y^R\mathbf{j} + M_z^R\mathbf{k}
\]

\( R_x, R_y, R_z \): the sums of \( x, y, z \) components of the given force

\( M_x^R, M_y^R, M_z^R \): the sums of the moments of the given force about the \( x, y, z \) axes

3.18 Equivalent Systems of Forces

two systems of forces are equivalent, therefore, if they may reduced to
the same force-couple system at given at point \( O \) of a rigid body

i.e. they are equivalent if, and only if, the sums of forces and the sums of
moments about a given point \( O \) are, respectively, equal
\[
\sum \mathbf{F} = \sum \mathbf{F}' \quad \rightarrow \quad \sum F_x = \sum F'_x, \quad \sum F_y = \sum F'_y, \quad \sum F_z = \sum F'_z, \\
\sum \mathbf{M}_0 = \sum \mathbf{M}_0' \quad \rightarrow \quad \sum M_x = \sum M'_x, \quad \sum M_y = \sum M'_y, \quad \sum M_z = \sum M'_z.
\]

3.19 Equipollent Systems of Vectors

Two systems of vectors satisfy the equations in the last section, the two systems are said to be equipollent.

If two systems of forces acting on a rigid body are equipollent, they are also equivalent.

Two systems of forces acting on different particles may be equipollent but not equivalent.

3.20 Further Reduction of a System of Forces

For a force-couple system, it can be reduced as \( \mathbf{R} \) and \( \mathbf{M}_0^R \) at \( O \).

When \( R = 0 \) \( \implies \) single couple vector \( \mathbf{M}_0^R \).

We now investigate the conditions under which a given system of forces may be reduced to a single force.

Two systems of forces which may be reduced to a single force, or resultant, are therefore the systems for which the force \( \mathbf{R} \) and the couple vector \( \mathbf{M}_0^R \) are mutually perpendicular.

This condition is generally not satisfied except for (1) concurrent forces, (2) coplanar forces, (3) parallel forces.

(1) Concurrent forces: they can be added directly into their resultant \( \mathbf{R} \).

(2) Coplanar forces: all forces lie on the \( x-y \) plane, the moment of each force about, \( \mathbf{M}_0^R \) will be perpendicular to the plane.
the force-couple system at \( O \) is
\[
R_x = \Sigma F_x, \quad R_y = \Sigma F_y, \quad M^R_z = M^R_0 = \Sigma M_0
\]
let the position of application is \( A(x, y) \), then
\[
x R_x - y R_x = M^R_0
\]
for \( y = 0 \) \( x = M^R_0 / R_y \)
for \( x = 0 \) \( y = M^R_0 / R_x \)

3) parallel forces: let all forces parallel to \( y \)-axis, of course \( R \) should be parallel to \( y \)-axis
\[
\therefore \text{ forces } / \parallel \text{ } y\text{-axis, hence}
\]
the moment about \( y \)-axis is zero, only \( M_x \) and \( M_z \) exist

move \( R \) to a new point of application \( A(x, 0, z) \) to reduced the force-couple system to a single force, then
\[
\mathbf{r} \times \mathbf{R} = M^R_0
\]
\[(x \mathbf{i} + z \mathbf{k}) \times R_y \mathbf{j} = M^R_x \mathbf{i} + M^R_z \mathbf{k}
\]
\[-z R_y \mathbf{i} + x R_y \mathbf{k} = M^R_x \mathbf{i} + M^R_z \mathbf{k}
\]
then \( x = M^R_x / R_y \) \( z = -M^R_z / R_y \)

in general case, the resultant \( R \) and the couple are not perpendicular, thus they cannot be reduced to a single force

resolving \( M^R_0 \) into components \( M_1 \) and \( M_2 \), where \( M_1 \) along the direction of \( R \) and \( M_2 \) is perpendicular to \( R \)

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move $\mathbf{R}$ to a new point of application $A$, such that the original system of force thus reduces to $\mathbf{R}$ and the couple $\mathbf{M}_1$, where $\mathbf{R}$ and $\mathbf{M}_1$ has the same direction, i.e. the couple acting in the plane perpendicular to $\mathbf{R}$

this combination of force-couple is called wrench, the line of action of $\mathbf{R}$ is known as the axis of the wrench

the magnitude of $\mathbf{M}_1$ is the projection of $\mathbf{M}_0^R$ on the line of action of $\mathbf{R}$, such that

$$M_1 = \mathbf{R} \cdot \mathbf{M}_0^R / |\mathbf{R}|$$

and the ratio $\frac{M_1}{|\mathbf{R}|} = \frac{\mathbf{R} \cdot \mathbf{M}_0^R}{\mathbf{R}^2} = p$ is called pitch of wrench

i.e.

$$M_1 = p \mathbf{R}$$

$$M_1 + \mathbf{r} \times \mathbf{R} = \mathbf{M}_0^R$$

$$p \mathbf{R} + \mathbf{r} \times \mathbf{R} = \mathbf{M}_0^R$$

if $\Sigma \mathbf{F} = \mathbf{R} = 0$, $\Sigma \mathbf{M}_0^R = 0$ (particular case), the rigid body is said to be equilibrium

Sample Problem 3.8

(a) equivalent force-couple at $A$
(b) equivalent force-couple at $B$
(c) a single force or resultant

(a) $\mathbf{R} = \Sigma \mathbf{F}$

$$= (150 - 600 + 100 - 250)j$$

$$= -600j \text{ (N)}$$

$M_A^R = \Sigma (\mathbf{r} \times \mathbf{F})$

$$= 1.6i \times (-600j) + 2.8i \times 100j + 4.8i \times (-250j)$$

$$= -1880k \text{ (N-m)}$$
(b) \[ M_B^R = M_A^R + BA \times R \]
\[ = -1880 \, k + (-4.8 \, i) \times (-600 \, j) \]
\[ = 1000 \, k \] (N-m)

(c) \[ r \times R = M_A^R \]
\[ x \, i \times (-600 \, j) = -1880 \, k \]
\[ x = 3.13 \, m \]

Sample Problem 3.9
determine the equivalent force and couple
resolve 125 N and 90 N forces in x
and y directions
125 N  \[ x\text{-dir.} = 125 \cos 40^\circ = 95.756 \, N \]
\[ y\text{-dir.} = -125 \sin 40^\circ = 80.348 \, N \]
90 N  \[ x\text{-dir.} = -45 \, N \]
\[ y\text{-dir.} = 77.942 \, N \]

\[ R_x = \Sigma F_x = 95.756 - 45 = 50.756 \, N \]
\[ R_y = \Sigma F_y = -80.348 - 200 - 77.942 = -358.29 \, N \]
\[ R = (50.756^2 + 358.29^2)^{\frac{1}{2}} = 361.87 \]

\[ \tan \theta = \frac{358.29}{50.756} \]
\[ \theta = 81.9^\circ \]
\[ R = 362 \, N \quad 81.9^\circ \]

\[ M_{eq} = -550 \times 90 \sin 35^\circ - 800 \times 200 \sin 65^\circ \]
\[ -1200 \times 125 \sin 65^\circ \]
\[ = -309 \, N\text{-m} \]
Sample Problem 3.10

replace the forces by a force-couple system at $A$

$B(75, 100, 50) \quad E(150, -50, 100)$

$BE = 75i - 150j + 50k \quad BE = 175$

$\lambda_{BE} = \frac{3}{7}i - \frac{6}{7}j + \frac{2}{7}k$

$F_B = 700 \lambda_{BE} = 300i - 600j + 200k$

$F_C = 707i - 707k$

$F_D = 600i + 1039j$

$R = \Sigma F = 1067i + 439j - 507k$

$AB = 0.075i + 0.05k \quad AC = 0.075i - 0.05k$

$AD = 0.1i - 0.1j$

$M_A^R = AB \times F_B + AC \times F_C + AD \times F_D$

$= \begin{vmatrix} i & j & k \\ 0.075 & 0 & 0.05 \\ 300 & -600 & 200 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0.075 & 0 & -0.05 \\ 707 & 0 & -707 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0.1 & -0.1 & 0 \\ 600 & 1039 & 0 \end{vmatrix}$

$= (30i - 45k) + (17.68j) + (163.9k)$

$= 30i + 17.68j + 118.9k \quad (N\cdot m)$

Sample Problem 3.11

determine the magnitude and point of application of the resultant $R$

$R = \Sigma F$

$M_0^R = \Sigma (r \times F)$

$r \ (m) \quad F \ (kN) \quad r \times F \ (kN\cdot m)$

$0 \quad -180j \quad 0$

$3i \quad -54j \quad -162k$

$3i + 1.5k \quad -36j \quad 54i - 108k$
1.2 \, i + 3 \, k \quad -90 \, j \quad 270 \, i - 108 \, k
\[ R = -360 \, j \quad M^R_0 = 324 \, i - 378 \, k \]
\[ r = x \, i + z \, k \]
\[ r \times R = M^R_0 \]
\[(x \, i + z \, k) \times (-360 \, j) = 324 \, i - 378 \, k \]
\[360 \, z \, i - 360 \, x \, k = 324 \, i - 378 \, k \]
\[ x = 1.05 \, m \quad z = 0.9 \, m \]

Sample Problem 3.12

replace the two forces into a wrench
determine \( R \), \( p \) and the intersection
of the wrench axis on \( y-z \) plane
\[ r_D = a \, j + a \, k \quad r_E = a \, i + a \, j \]

force-couple at \( O \)
\[ R = F_1 + F_2 = P \, (i + j) \]
\[ M^R_0 = r_E \times F_1 + r_D \times F_2 \]
\[ = a \, P \, (i + j) \times i + a \, P \, (j + k) \times j \]
\[ = -a \, P \, (i + k) \]

pitch \( p = \frac{R \cdot M^R_0}{R^2} = \frac{P \, (i + j) \cdot [-a \, P \, (i + k)]}{(\sqrt{2} \, P)^2} \]
\[ = -\frac{a \, P^2}{2 \, P^2} = -\frac{a}{2} \]
\[ M_1 = p \, R = -a \, P \, (i + k) / 2 \]
\[ M_1 + r \times R = M^R_0 \]
\[ r = y \, j + z \, k \quad \text{(on } y-z \text{ plane)} \]
\[-a \, P \, (i + k) / 2 + (y \, j + z \, k) \times P \, (i + j) = -a \, P \, (i + k) \]
\[(-a \, P / 2 - P \, z) \, i - (a \, P / 2 - P \, z) \, j - P \, y \, k = -a \, P \, (i + k) \]
\[\Rightarrow y = a \quad z = a / 2 \quad \text{the position of } G \text{ is } G(0, a, a/2)\]