CHAPTER 8

NATURAL AND STEP RESPONSES OF RLC CIRCUITS

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8.1 Linear Second Order Circuits

- Circuits containing two energy storage elements.
- Described by differential equations that contain second order derivatives.
- Need two initial conditions to get the unique solution.

Examples
(a) RLC parallel circuit
(b) RLC series circuit
8.1 Linear Second Order Circuits

(c) 2L+R, RL circuit

(d) 2C+R, RC circuit

8.2 Solution Steps

Step 1: Choose nodal analysis or mesh analysis approach

Step 2: Differentiate the equation as many times as required to get the standard form of a second order differential equation.

\[ a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + x = y(t) \]
8.2 Solution Steps

Step 3 : Solving the differential equation
(1) homogeneous solution \( x_h(t) \)
(2) particular solution \( x_p(t) \)
\[ x(t) = x_h(t) + x_p(t) \]

Step 4 : Find the initial conditions
\( x(0^+) \) and \( \frac{d}{dt} x(0^+) \) and then get the unique solution

8.3 Finding Initial Values

Under DC steady state, L is like a short circuit and C is like an open circuit.
8.3 Finding Initial Values

Under transient condition, L is like an open circuit and C is like a short circuit because $i_L(t)$ and $v_C(t)$ are continuous functions if the input is bounded.

![Diagram of an inductor and capacitor in series with initial conditions.]

8.3 Finding Initial Values

Under transient condition, L is like an open circuit and C is like a short circuit because $i_L(t)$ and $v_C(t)$ are continuous functions if the input is bounded.

![Diagram of a capacitor with initial conditions.]

C.T. Pan
8.3 Finding Initial Values

To find $\frac{di_L(0^+)}{dt}$ and $\frac{dv_C(0^+)}{dt}$, use the following relations:

\[ L \frac{di_L(0^+)}{dt} = v_L(0^+) \Rightarrow \frac{d}{dt} i_L(0^+) = \frac{v_L(0^+)}{L} \]
\[ C \frac{dv_C(0^+)}{dt} = i_c(0^+) \Rightarrow \frac{d}{dt} v_C(0^+) = \frac{i_C(0^+)}{C} \]

One can find $v_L(0^+)$ and $i_c(0^+)$ using either nodal or mesh analysis.

Example 1

The circuit is under steady state. The switch is opened at $t = 0$.

Find: (a) $i(0^+), \frac{d}{dt} i(0^+)$
(b) $v(0^+), \frac{d}{dt} v(0^+)$
(c) $i(\infty), v(\infty)$
8.3 Finding Initial Values

Example 1 (cont.)

\[ t < 0 \]

\[ i(0^-) = 2 \text{ A} \]

\[ v(0^-) = 4 \text{ V} \]

\[ i(0^+) = i(0^-) = 2 \text{ A} \]

\[ v(0^+) = v(0^-) = 4 \text{ V} \]

\[ t = 0^+ \]

\[ Q_L \frac{di}{dt} = V_L \quad \therefore \quad \frac{d}{dt}i(0^+) = \frac{V_L(0^+)}{L} \]

\[ Q_C \frac{dv}{dt} = i_C \quad \therefore \quad \frac{d}{dt}v(0^+) = \frac{i_C(0^+)}{C} \]
8.3 Finding Initial Values

Example 1 (cont.)

\[ KVL: \quad 2A \times 4 + v_L(0^+) + 4V = 12V \]
\[ \therefore v_L(0^+) = 0 \]
\[ \therefore \frac{d}{dt} i(0^+) = 0 \]

\[ KCL: \quad i_c(0^+) = 2A \]
\[ \therefore \frac{d}{dt} v(0^+) = 0 \]
\[ \therefore \frac{i_c(0^+)}{C} = \frac{2}{0.1} = 20 \text{ V/S} \]

\[
\begin{array}{c}
\begin{array}{c}
4 \Omega \\
2A \\
20 \Omega \\
0.25H \\
12V \\
- \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
4 \Omega \\
20 \Omega \\
0.1F \\
12V \\
- \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
L \text{ is short circuit} \quad t \to \infty \\
C \text{ is open} \\
\therefore i(\infty) = 0 \\
v(\infty) = 12V
\end{array}
\end{array}
\]
8.3 Finding Initial Values

Example 2

Find:
(a) $i_L(0^+)$, \( \frac{d}{dt}i_L(0^+)\), \(i_L(\infty)\)
(b) $v_C(0^+)$, \( \frac{d}{dt}v_C(0^+)\), \(v_C(\infty)\)
(c) $v_R(0^+)$, \( \frac{d}{dt}v_R(0^+)\), \(v_R(\infty)\)

Example 2 (cont.)

\[ i_L(0^-) = 0 \]
\[ v_C(0^-) = -20V \]
\[ v_R(0^-) = 0 \]
8.3 Finding Initial Values

Example 2 (cont.)

\[ t = 0^+ \]

\[ i_L(0^+) = i_L(0^-) = 0 \]
\[ v_c(0^+) = v_c(0^-) = -20V \]

\[ i_{2Ω}(0^+) = 3A \times \frac{4}{2+4} = 2A \]
\[ v_{R}(0^+) = 2A \times 2 = 4V \]
\[ i_c(0^+) = 3A \times \frac{2}{2+4} = 1A \]
\[ \therefore \frac{d}{dt} i_L(0^+) = \frac{v_L(0^+)}{L} = 0 = 0 \]
\[ \frac{d}{dt} v_c(0^+) = \frac{i_c(0^+)}{c} = \frac{1}{\frac{1}{2}} = 2 \frac{V}{S} \]
\[ \frac{d}{dt} v_R(0^+) = ? \]
8.3 Finding Initial Values

Example 2 (cont.)

From KVL: \(-v_R + v_o + v_C + 20 = 0\)

Take derivative

\[
\frac{d}{dt}v_R(t) = \frac{d}{dt}v_o(t) + \frac{d}{dt}v_C(t)
\]

\[
\therefore \frac{d}{dt}v_R(0^+) = \frac{d}{dt}v_o(0^+) + \frac{d}{dt}v_C(0^+)L \ (A)
\]

Also, from KCL: \(3A = \frac{v_R(t)}{2} + \frac{v_o(t)}{4}\)

Take derivative

\[
0 = \frac{1}{2} \frac{d}{dt}v_R(t) + \frac{1}{4} \frac{d}{dt}v_o(t)L \ (B)
\]
8.3 Finding Initial Values

Example 2 (cont.)

From (B) \( \frac{d}{dt}v_c(0^+) = -2\frac{d}{dt}v_R(0^+) \) \( L \) \( L \) (C)

From (A) and (C)

\[ \frac{d}{dt}v_R(0^+) = -2\frac{d}{dt}v_R(0^+) + 2 \]

\[ \therefore \frac{d}{dt}v_R(0^+) = 2 \frac{V}{3/S} \]

\[ \therefore i_L(\infty) = 3A \times \frac{2}{2+4} = 1A \]

\[ v_c(\infty) = -20V \]

\[ v_R(\infty) = 3A \times (2\Omega \ P 4\Omega) = 4V \]
8.4 The Natural Response of a Series/Parallel RLC Circuit

(a) The source-free series RLC circuit

This section is an important background for studying filter design and communication networks.

\[ \begin{aligned}
  \text{initial conditions} \\
  i(0) &= I_0 \\
  v(0) &= V_0
\end{aligned} \]

8.4 The Natural Response of a Series/Parallel RLC Circuit

\[ \begin{aligned}
  \text{initial conditions} \\
  i(0) &= I_0 \\
  v(0) &= V_0
\end{aligned} \]

Step 1: Mesh analysis

\[ Ri + L \frac{di}{dt} + \frac{1}{C} \int_i dt = 0 \]

To eliminate the integral, take derivative

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = 0 \]
8.4 The Natural Response of a Series/Parallel RLC Circuit

**Step 2: Homogeneous Solution, Characteristic Equation**

\[ S^2 + \frac{R}{L}S + \frac{1}{LC} = 0 \]

\[ S^2 + 2\alpha S + \omega_0^2 = 0 \]

\[ \Rightarrow \omega_0 = \sqrt{\omega_0^2} \]

\[ \omega_0 : \text{undamped resonant frequency (rad/s)} \]

\[ \alpha : \text{damping factor or neper frequency} \]

**Characteristic Roots (Natural Frequencies)**

\[ S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]

\[ S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\[ \therefore i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} \]

**Need Two Initial Conditions, i.e.,**

\[ i(0) = I_0 \]

\[ \frac{di}{dt}(0) = \frac{d}{dt}i(0^+) \]
8.4 The Natural Response of a Series/Parallel RLC Circuit

**Case 1 Overdamped Case (α > ω₀)**
- Two real roots
- \( i(t) = A_1 e^{\alpha t} + A_2 e^{\beta t} \)

**Case 2 Critically Damping Case (α = ω₀)**
- Equal real roots
- \( S_1 = S_2 = -\frac{R}{2L} = -\alpha \)
- \( i(t) = (A_1 + A_2 t) e^{-\alpha t} \)

**Case 3 Underdamped Case (α < ω₀)**
- Complex conjugate roots
- \( S_1 = -\alpha + j\omega_d \)
- \( S_2 = -\alpha - j\omega_d \)
- \( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \) damping frequency
- \( i(t) = e^{-\alpha t} \left( B_1 \cos \omega_d t + B_2 \sin \omega_d t \right) \)

Once \( i(t) \) is obtained, solutions of other variables can be obtained from this mesh current.
The damping effect is due to the presence of resistance $R$. The damping factor $\alpha$ determines the rate at which the response is damped.

If $R=0$, the circuit is said to be lossless and the oscillatory response will continue.

The damped oscillation exhibited by the underdamped response is known as ringing. It stems from the ability of the $L$ and $C$ to transfer energy back and forth between them.

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8.4 The Natural Response of a Series/Parallel RLC Circuit

**Step 3: Initial Condition**

$t = 0^+$, $i(0^+) = i(0^-) = I_o$

From mesh equation, let $t = 0^+$

$$Ri(0^+) + L\frac{d}{dt}i(0^+) + \frac{1}{C}\int_{\infty}^{0^+} i(t)dt = 0$$

$$\therefore \quad \frac{d}{dt}i(0^+) = -\frac{R}{L}i(0^+) - \frac{V_o}{L} = -\frac{R}{L}I_o - \frac{V_o}{L}$$
8.4 The Natural Response of a Series/Parallel RLC Circuit

or from equivalent circuit at $t = 0^+$

$$L \frac{d}{dt} i(0^+) = v_L = -(I_o R + V_0)$$

∴ $\frac{d}{dt} i(0^+) = - \frac{I_o R}{L} - \frac{V_0}{L}$

(b) The source-free parallel RLC circuit

Step 1 : Nodal Equation

$$\frac{v}{R} + \frac{1}{L} \int v dt + C \frac{dv}{dt} = 0$$

Taking derivative to eliminate the integral

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0$$

initial inductor current $I_o$

initial capacitor voltage $V_o$
8.4 The Natural Response of a Series/Parallel RLC Circuit

Step 2: Homogeneous solution

Characteristic equation

\[ S^2 + \frac{1}{RC}S + \frac{1}{LC} = 0 \]

\[ S^2 + 2\alpha S + \omega_0^2 = 0 \]

\[ \Rightarrow \alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{LC} \]

Characteristic roots (natural frequencies)

\[ S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]

\[ S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

CASE 1. Overdamped Case (\( \alpha > \omega_0 \))

\[ v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} \]

CASE 2. Critically Damped Case (\( \alpha = \omega_0 \))

\[ v(t) = (A_1 + A_2 t) e^{-\alpha t} \]

CASE 3. Underdamped Case (\( \alpha < \omega_0 \))

\[ v(t) = e^{-\alpha t} \left( A_1 \cos w_d t + A_2 \sin w_d t \right) \]
8.4 The Natural Response of a Series/Parallel RLC Circuit

Step 3: Initial Condition

\[ v(0^+) = i(0^+) = V_0 \]

From nodal equation

\[ \frac{v(0^+)}{R} + \frac{1}{L} \int_{-\infty}^{0^+} v dt + C \frac{d}{dt} v(0^+) = 0 \]

\[ \therefore C \frac{d}{dt} v(0^+) = -\frac{v(0^+)}{R} - \frac{1}{L} \int_{-\infty}^{0^+} v dt \]

\[ = -\frac{V_0}{R} - I_0 \]

\[ \therefore \frac{d}{dt} v(0^+) = -\frac{V_0}{RC} - \frac{I_0}{C} \]

Once the nodal voltage is obtained, any other unknown of the circuit can be found.

or from equivalent circuit at \( t = 0^+ \)

\[ C \frac{d}{dt} i_c(0^+) = V_0 \]

\[ \therefore \frac{d}{dt} i_c(0^+) = \frac{i_c(0^+)}{C} \]

From KCL, \( i_c(0^+) = -I_0 - \frac{V_0}{R} \)
8.5 The Step Response of a Series/Parallel RLC Circuit

(a) Step response of a series RLC circuit

Step 1. Mesh analysis \( i = i_L \)

\[
L \frac{di}{dt} + Ri + \frac{1}{C} \int idt = V_s, \quad t > 0
\]

Case (i) take derivative

\[
L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = 0
\]

Same as natural response but with \( i(0^+)=I_0 \)

\[
\frac{d}{dt} i(0+) = \frac{V_s - I_0 R - V_0}{L}
\]

Case (ii) use \( v \) as unknown

\[
i = C \frac{dv}{dt}
\]

\[
d^2v + \frac{R}{L} \frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}
\]

Step 2. Complete solution = \( v_h + v_p \)

\[
v_p(t) = V_s
\]

\[
v_h(t) = \begin{cases} A e^{s_1} + A_1 e^{s_2} & \text{overdamped} \\ (A_1 + A_2 t) e^{-\alpha t} & \text{critically damped} \\ e^{-\alpha t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t) & \text{underdamped} \end{cases}
\]
8.5 The Step Response of a Series/Parallel RLC Circuit

Step 3. Initial conditions

\[ v(0^+) = v(0^-) = V_0 \]
\[ \frac{d}{dt}v(0^+) = ? \]
\[ C \frac{dv}{dt} = i_c \]

\[ \therefore \frac{d}{dt}v(0^+) = \frac{i_c(0^+)}{C} = \frac{I_0}{C} \]

Then the unique solution can be determined.

---

8.5 The Step Response of a Series/Parallel RLC Circuit

(b) Step response of a parallel RLC circuit

Step 1. Nodal equation

\[ \frac{\nu}{R} + \frac{1}{L} \int vdt + C \frac{dv}{dt} = I_s , t > 0 \]

Case (i) Take derivative

\[ C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{1}{L} v = 0 \]

Same as natural response.
8.5 The Step Response of a Series/Parallel RLC Circuit

Case (ii) Let

\[ v = L \frac{di}{dt} \Rightarrow \frac{d^2i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{1}{LC} i = \frac{I_s}{LC}, \quad t > 0 \]

Step 2. Complete solution = \( i_h(t) + i_p(t) \)

\[
\begin{align*}
    i_p(t) &= I_s, \\
    i_h(t) &= \begin{cases} 
        A_1 e^{at} + A_2 e^{bt} & \text{overdamped} \\
        (A_1 + A_2 t) e^{-\alpha t} & \text{critically damped} \\
        e^{-\alpha t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t) & \text{underdamped}
    \end{cases}
\end{align*}
\]

Step 3. Initial Condition

\[
\begin{align*}
    i(0^-) &= i(0^+) = I_o, \\
    L \frac{di}{dt} &= v_L \Rightarrow \frac{d}{dt} i(0^+) = \frac{v_L(0^+)}{L} = \frac{V_o}{L}
\end{align*}
\]

Example

\[
\begin{align*}
  t < 0, \quad i(0^-) &= 0, \quad v(0^-) = 12V \\
  t > 0
\end{align*}
\]
Method (a)

From KCL \( i = \frac{v}{2} + \frac{1}{2} \frac{dv}{dt} \) \hspace{1cm} (A)

From KVL \( 4i + \frac{1}{dt} + v = 12 \) \hspace{1cm} (B)

Substitute (A) into (B)

\[
(2v + 2 \frac{dv}{dt}) + (\frac{1}{2} \frac{dv}{dt} + \frac{1}{2} \frac{d^2v}{dt^2}) + v = 12
\]

\[
\Rightarrow \frac{d^2v}{dt^2} + 5 \frac{dv}{dt} + 6v = 24V
\]

**Characteristic equation**

\( (s + 2)(s + 3) = 0 \)

\( s_1 = -2, s_2 = -3 \)

---

8.5 The Step Response of a Series/Parallel RLC Circuit

\( v_h(t) = A_1 e^{-2t} + A_2 e^{-3t} \)

\( v_p(t) = \frac{24}{6} = 4V \)

\( \therefore v(t) = 4 + A_1 e^{-2t} + A_2 e^{-3t} \)

Initial Condition \( v(0^+) = v(0^-) = 12V \)

\( t=0^+ \)


8.5 The Step Response of a Series/Parallel RLC Circuit

\[ \therefore 4 + A_1 + A_2 = 12 \]
\[ -2A_1 - 3A_2 = -12 \]
\[ \therefore A_1 = 12, \ A_2 = -4 \]
\[ \therefore v(t) = 4 + 12e^{-2t} - 4e^{-3t}, t \geq 0 \]

Method (b) Using Mesh Analysis

\[ t > 0 \]

\[ \begin{align*}
\text{Method (b) Using Mesh Analysis} \\
4 \Omega & \quad i \\
1 \text{H} & \quad 2 \Omega \\
12 \text{V} & \quad \text{v} \\
\ \ \ & \quad \frac{1}{2} \text{F} \\
\end{align*} \]

8.5 The Step Response of a Series/Parallel RLC Circuit

\[ \frac{1}{2} \frac{d^2 i}{dt^2} + 4i + (i_1 - i_2)2 = 12 \quad \text{......(A)} \]
\[ (i_2 - i_1) \times 2\Omega + \frac{1}{1/2} \int i_2 \, dt = 0 \quad \text{......(B)} \]

From (B),
\[ 2 \frac{d^2 i_2}{dt^2} - 2 \frac{di_1}{dt} + 2i_2 = 0 \]
\[ \frac{di}{dt} + 6i - 2i_2 = 12 \]
\[ -2 \frac{di_1}{dt} + 2 \frac{di_2}{dt} + 2i_2 = 0 \]
8.5 The Step Response of a Series/Parallel RLC Circuit

In matrix form with operator $D \frac{d}{dt}$

\[
\begin{bmatrix}
D + 6 & -2 \\
-2D & 2D + 2
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
0
\end{bmatrix} \quad (C)
\]

Initial Condition, $i_1(0^+) = i(0^+) = i(0^-) = 0A$

- $i_2(0^+) = i_c(0^+) = -6A$
- $\frac{di_1(0^+)}{dt} = \frac{v_c(0^+)}{L} = 0$
- $\frac{di_2(0^+)}{dt} = 0A/S$

Eliminate $i_2$ variable from (C)

\[
\frac{d^2i_1}{dt^2} + 5 \frac{di_1}{dt} + 6i_1 = 12
\]

$\therefore i_1(t) = 2 - 6e^{-2t} + 4e^{-3t}, t \geq 0$

Or eliminate $i_1$ variable from (C)

\[
\frac{d^2i_2}{dt^2} + 5 \frac{di_2}{dt} + 6i_2 = 0
\]

$\therefore i_2(t) = -12e^{-2t} + 6e^{-3t}, t \geq 0$
Problem: (a) Time consuming to eliminate the other variable to get a higher order differential equation.
(b) It is also necessary to obtain the desired initial conditions.
(c) As the order gets higher when the network contains more energy storage elements, the process gets more complicated.

The difficulty can be overcome by using state equation formulation.

8.6 State Equation

When the differential equations of a circuit is written in the following form:

\[
\frac{d}{dt}x = f(x, u, t)\\
\begin{align*}
\begin{bmatrix}
\ddots
\end{bmatrix} & \text{state vector} \\
\begin{bmatrix}
\ddots
\end{bmatrix} & \text{input vector} \\
\begin{bmatrix}
\ddots
\end{bmatrix} & \text{vector function}
\end{align*}
\]

\[
\begin{bmatrix}
\ddots
\end{bmatrix} = \begin{bmatrix}
    x_1 & x_2 & \ldots & x_n
\end{bmatrix} \\
\begin{bmatrix}
\ddots
\end{bmatrix} = \begin{bmatrix}
    u_1 & u_2 & \ldots & u_m
\end{bmatrix} \\
\begin{bmatrix}
\ddots
\end{bmatrix} = \begin{bmatrix}
    f_1 & f_2 & \ldots & f_n
\end{bmatrix}
\]
8.6 State Equation

It is said that the circuit equations are in the state equation form.

(a) This form lends itself most easily to analog or digital computer programming.

(b) The extension to nonlinear and/or time varying networks is quite easy.

(c) In this form, a number of theoretic concepts of systems are readily applicable to networks.

For a linear time-invarying circuit, a simpler form

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu, \quad \text{state equation} \\
\gamma &= Cx + Du, \quad \text{output equation}
\end{align*}
\]

A : \(n \times n\) matrix.
B : \(n \times m\) matrix.
C : \(l \times n\) matrix.
D : \(l \times m\) matrix.

Note that on the right hand side of the state or output equation, only \(x\) and \(u\) are allowed.
8.6 State Equation

Step 1. Pick a tree which contains all the capacitors and none of inductors.
Step 2. Use the tree-branch capacitor voltages and the link inductor currents as unknown (i.e., state) variables.
Note: (a) Nodal Analysis
   Every unknown of the circuit can be calculated from nodal voltages.
(b) Mesh Analysis
   Every unknown of the circuit can be calculated from mesh currents.

(c) State Equation
   - The chosen variables include both voltage and current unknown. It belongs to mixed type.
   - Every unknown of the circuit can be calculated from the state variables by replacing each inductor with a current source and each capacitor with a voltage source and then solving the resulting resistive circuit.
8.6 State Equation

Step 3. Write a fundamental cutset equation (i.e. KCL equation) for each capacitor. Note that in these cutset equations, all branch currents must be expressed in terms of x and u.

Step 4. Write a fundamental loop equation (i.e. KVL equation) for each inductor. Note that in these loop equations, all branch voltages must be expressed in terms of x and u.

Step 5. Rearrange the above equations into standard form and find the solution for the given initial condition.

Example 1

Step 1
8.6 State Equation

Step 2: Choose \( i \) and \( v \) as state variables.

Step 3: Fundamental cutset about the capacitor tree branch.

\[
C \frac{dv}{dt} = i - \frac{v}{2}
\]

Step 4: Fundamental loop for the inductor link.

\[
L \frac{di}{dt} + v - 12V + 4i = 0
\]

Step 5: The desired solutions are \( v \) and \( i \) with initial condition

\[
[v(0^+)] = 12V
\]

\[
i(0^+) = 0A
\]

\[
\begin{bmatrix}
\frac{d}{dt} \\
\end{bmatrix}
\begin{bmatrix}
v \\
i
\end{bmatrix}
= 
\begin{bmatrix}
1 & -1 \\
-4 & -1
\end{bmatrix}
\begin{bmatrix}
v \\
i
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1/L
\end{bmatrix}
\]

\[
\begin{bmatrix}
y \\
i
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v \\
i
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
8.6 State Equation

Example 2

\[
\frac{d^3v}{dt^3} + 5 \frac{d^2v}{dt^2} + 4 \frac{dv}{dt} + 3v = u(t)
\]

Let

\[
\begin{align*}
  x_1 &= v(t) \\
  x_2 &= \frac{dv(t)}{dt} \\
  x_3 &= \frac{d^2v(t)}{dt^2}
\end{align*}
\]

\[
\begin{align*}
  \frac{dx_1}{dt} &= x_2 \\
  \frac{dx_2}{dt} &= x_3 \\
  \frac{dx_3}{dt} &= \frac{d^3v}{dt^3}
\end{align*}
\]

\[
\begin{align*}
  &= -5 \frac{d^2v}{dt^2} - 4 \frac{dv}{dt} - 3v + u(t) \\
  &= -5x_3 - 4x_2 - 3x_1 + u(t)
\end{align*}
\]

8.6 State Equation

State space representation

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -3 & -4 & -5
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix} u(t)
\]

\[
y = \begin{bmatrix}
  1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
+ \begin{bmatrix}
  0
\end{bmatrix} u(t)
\]

A high order differential equation can be represented in the form of state equation.
8.6 State Equation

Example 3: Find $v_{R4}$

There are 8 nodes.

8.6 State Equation

Step 1: Pick a tree as follows:

There are 7 tree branches and 3 links.
8.6 State Equation

Step 2: Choose \(i_{L1}, i_{L2}, v_{C1}, v_{C2}\) as unknowns.

Step 3: Fundamental cutsets (KCL) for capacitors.

\[
\begin{align*}
C_1 \frac{dv_{C1}}{dt} &= i_{L1} \\
C_2 \frac{dv_{C2}}{dt} &= i_{L1} + i_{L2}
\end{align*}
\]

Step 4: Fundamental loops (KVL) for inductors.

\[
\begin{align*}
L_1 \frac{di_{L1}}{dt} &= -v_{R1} - v_{C1} - v_{C2} - v_{R5} + v_{R4} \\
&= -R_1 i_{L1} - v_{C1} - v_{C2} - R_5 (i_{L1} + i_{L2}) + v_{R4}
\end{align*}
\]

Note that \(v_{R4}\) should be expressed in terms of \(x\) and \(u\).

8.6 State Equation

Absorb voltages \(v_{R1}\) and \(v_{C1}\) in current source \(i_{L1}\), and \(v_{R2}\) in \(i_{L2}\).

Absorb voltages \(v_{C2}\) and \(v_{R5}\) in \((i_{L1} + i_{L2})\) current source.
8.6 State Equation

\[ v_{R4} = e_s \frac{R_4}{R_3 + R_4} - (i_{L1} + i_{L2}) \frac{R_3 R_4}{R_3 + R_4} \]

**Step 5**

\[
\begin{bmatrix}
\frac{dv_{C1}}{dt} \\
\frac{dv_{C2}}{dt} \\
\frac{di_{L1}}{dt} \\
\frac{di_{L2}}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \frac{1}{C_1} & 0 \\
0 & 0 & \frac{1}{C_2} & \frac{1}{C_2} \\
-\frac{1}{L_1} & -\frac{1}{L_1} & \frac{R_3 + R_2}{L_1} & -\frac{R}{L_1} \\
0 & -\frac{1}{L_2} & -\frac{R}{L_2} & -\frac{(R_2 + R)}{L_2}
\end{bmatrix}
\begin{bmatrix}
v_{C1} \\
v_{C2} \\
i_{L1} \\
i_{L2}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\frac{1}{L_1} \\
\frac{1}{L_2}
\end{bmatrix}
+ R_4 \frac{v_{R4}}{R_3 + R_4} e_s
\]

\[ v_{R4} =
\begin{bmatrix}
0 & 0 & -\frac{R_3 R_4}{R_3 + R_4} & -\frac{R_3 R_4}{R_3 + R_4} \\
-\frac{R_3 R_4}{R_3 + R_4} & \frac{R_3 R_4}{R_3 + R_4} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_{C1} \\
v_{C2} \\
i_{L1} \\
i_{L2}
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{R_4}{R_3 + R_4} e_s
\end{bmatrix}
\]

where \( R \left( R_3 + R_4 \right) \)
### 8.6 State Equation

#### Special case

**(a)**

From KCL:

\[ i_1 + i_2 + i_3 = 0 \]

\[ \therefore i_3 = -i_1 - i_2 \]

Inductor current \( i_3 \) is dependent on \( i_1 \) and \( i_2 \) and is no longer a state variable.

One can choose only \( n-1 \) (here 2) inductor currents as state variables.

---

#### (b)

From KVL:

\[ v_{C1} + v_{C2} = v_{C3} \]

One can choose \( n-1 \) (here 2) capacitor voltages as state variables.
Objective 1: Be able to find the initial values and the initial derivative values.

Objective 2: Be able to determine the natural response and the step response of a series RLC circuit.

Objective 3: Be able to determine the natural response and the step response of a parallel RLC circuit.

Objective 4: Be able to obtain the state equation and output equation of a linear circuit.