CHAPTER 7

RESPONSE OF FIRST-ORDER RC AND RL CIRCUITS

CONTENTS

7.1 The Natural Response of an RC Circuit
7.2 The Natural Response of an RL Circuit
7.3 Singularity Functions
7.4 The Step Response of RC and RL Circuit
7.1 The Natural Response of an RC Circuit

Resistive Circuit => RC Circuit

algebraic equations => differential equations

Same Solution Methods (a) Nodal Analysis
    (b) Mesh Analysis

The solution of a linear circuit, called dynamic response, can be decomposed into

Natural Response + Forced Response

or in the form of

Steady Response + Transient Response
7.1 The Natural Response of an RC Circuit

- The natural response is due to the initial condition of the storage component (C or L).
- The forced response is resulted from external input (or force).
- In this chapter, a constant input (DC input) will be considered and the forced response is called step response.

Example 1: Two forms of the first order circuit for natural response

Find $v(t)$

Find $i(t)$
Example 1: (cont.) Four forms of the first order circuit for step response

A capacitor connected to a Thevenin equivalent

A capacitor connected to a Norton equivalent

An inductor connected to a Thevenin equivalent

An inductor connected to a Norton equivalent
7.1 The Natural Response of an RC Circuit

Example 2

\[ t \geq 0 \]

\[
\begin{align*}
\text{v} + & \quad \text{i}_c \\
\text{i}_c & \quad \text{R} \\
\text{v} & \quad \text{R} \\
\text{v} + & \quad \text{i}_c \\
\text{i}_c & \quad \text{R}
\end{align*}
\]

nodal analysis

\[
i_c + i_R = 0
\]

\[
C \frac{dv}{dt} + \frac{1}{R} v = 0
\]

7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

characteristic root \( S \),

\[
CS + \frac{1}{R} = 0
\]

\[
S = -\frac{1}{RC}
\]

\[
\therefore v(t) = Ke^{-\frac{t}{RC}}, t \geq 0
\]

From the initial condition

\[
v(0^+) = v(0^-) = V_0
\]

\[
\therefore v(t) = V_0e^{-\frac{t}{RC}}
\]
Example 2 (cont.)

\[ \tau = RC \quad \text{time constant} \]

\[ v(t) = V_0 e^{-\frac{t}{\tau}}, t \geq 0 \]

\[ t = \tau, \quad \frac{v(t)}{V_0} = 0.36788 \]

\[ t = 3\tau, \quad 4.979\% < 5\% \]

\[ t = 5\tau, \quad 0.674\% < 1\% \]

It is customary to assume that the capacitor is fully discharged after five time constants.

\[ i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}}, \quad t \geq 0 \]

The power dissipated in R is

\[ p(t) = vi_R = \frac{V_0^2}{R} e^{-\frac{2t}{\tau}} \]
7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

The energy dissipated in R is

\[
\begin{align*}
\mathcal{W}_R(t) &= \int_0^t P dt = \frac{1}{2} CV_0^2 (1 - e^{-\frac{2t}{\tau}}) \\
\mathcal{W}_R(\infty) &= \frac{1}{2} CV_0^2
\end{align*}
\]

Example 3: Find \(v_c\), \(v_x\), and \(i_x\) for \(t > 0\)

This circuit contains only one energy storage element.
Step 1. Use Thevenin theorem to find the equivalent \(R_{\text{TH}}\) looking into a-b terminals.
The Natural Response of an RC Circuit

Example 3: Find $v_c$, $v_x$, and $i_x$ for $t > 0$

\[
\begin{align*}
\text{Step 1.} & \quad R_{TH} = (8 + 12) \Omega = 20 \Omega \\
\text{Step 2.} & \quad v_c(0) = 15V \\
\text{Step 2.} & \quad 0.1 \frac{dv_c}{dt} + \frac{v_c}{4} = 0 \\
& \text{where } \tau = RC = 0.4 \text{ s} \\
& \quad v_c(t) = 15e^{-2.5t} V, t \geq 0
\end{align*}
\]
7.1 The Natural Response of an RC Circuit

Step 3.

By using voltage divider principle

\[ v_c(t) = \frac{12}{8+12} v_x(t) \]
\[ = 9e^{-0.75t} V, t \geq 0 \]
\[ i_c(t) = \frac{v_c(t)}{12} = 0.75e^{-0.75t} A \]

7.2 The Natural Response of an RL Circuit

Example 4

Remember that \(v_c(t)\) and \(i_L(t)\) are continuous functions for bounded inputs.

\[ t \leq 0, \quad i = \frac{V_s}{R_s} = i(0^-) \]
7.2 The Natural Response of an RL Circuit

\[ t \geq 0 \]

\[ \begin{align*}
- & \quad \begin{align*}
L & \quad i \quad R \\
\quad i & \quad v_L \\
\quad i & \quad v_R
\end{align*}
\end{align*} \]

**mesh analysis**

\[ v_L + v_R = 0 \]

\[ L \frac{di}{dt} + Ri = 0 \]

\[ \frac{L}{R} \quad \text{time constant} \]

- \[ \frac{R}{L} \]

**characteristic equation**

\[ LS + R = 0 \]

\[ \therefore S = - \frac{R}{L} \]

\[ i(t) = Ke^{\frac{R}{L}t}, t \geq 0 \]

\[ i(0^+) = i(0^-) = I_0 \]

\[ \therefore i(t) = I_0 e^{\frac{R}{L}t}, t \geq 0 \]
7.2 The Natural Response of an RL Circuit

\[ v_R(t) = Ri = I_0 R e^{-\frac{t}{\tau}} \]

\[ p_R = v_R i = I_0^2 R e^{-\frac{2t}{\tau}} \]

\[ w_R(t) = \int_0^t p_R(t) dt = \frac{1}{2} LI_0^2 (1 - e^{-\frac{2t}{\tau}}) \]

\[ w_R(\infty) = \frac{1}{2} LI_0^2 \]

Example 5: \( i(0) = 10A \), find \( i(t) \) and \( i_x(t) \)
7.2 The Natural Response of an RL Circuit

Step 1. Find the Thevenin equivalent circuit looking into a-b terminals.

Apply $i_s$, find $e$. Then $R_{TH} = e / i_s$

$$i = -i_s$$

$$\frac{e - 3i}{4} + \frac{e}{2} = i_s$$

$$\frac{e}{4} + \frac{3}{4}i_s + \frac{e}{2} = i_s$$

Step 2.

$$3 \cdot \frac{e}{4} = \frac{1}{3}i_s$$

$$\therefore \frac{e}{i_s} = \frac{1}{3} \Omega$$

Mesh equation

$$0.5 \frac{di}{dt} + \frac{1}{3}i = 0$$

$$\tau = \frac{L}{R} = \frac{0.5}{1/3} = 1.5 \text{ S}$$

$$\therefore i(t) = 10e^{\frac{2}{3}t} \text{ A, } t \geq 0$$
7.2 The Natural Response of an RL Circuit

Step 3 Replace the inductor with an equivalent current source with $i(t)$ and solve the resistive circuit.

For this problem, since $2\Omega$ is in parallel with the inductor, it is trivial to get

\[
\nu_L(t) = L \frac{di}{dt} \\
= 0.5 \frac{d}{dt} (10 e^{\frac{2}{3}t}) \\
= 0.5 \times 10 \times \left( -\frac{2}{3} \right) e^{\frac{2}{3}t} \\
= -\frac{10}{3} e^{\frac{2}{3}t} \text{ V} \\
\therefore \ i_x = \frac{\nu_L}{2\Omega} = -\frac{5}{3} e^{\frac{2}{3}t} \text{ V, } t \geq 0
\]
7.2 The Natural Response of an RL Circuit

Example 6: The switch has been closed for a long time. At $t=0$, it is opened. Find $i_x$.

$$i_i(0^-) = \frac{40V}{2+(12/4)} = 8 \text{ A}$$

$$i_x(0^-) = 0$$

$$i(0^+) = i_i(0^-) \times \frac{12}{12+4} = 8 \times \frac{3}{4} = 6 \text{ A}$$

$\therefore$ Initial condition $i(0^-) = i(0^+) = 6 \text{ A}$
7.2 The Natural Response of an RL Circuit

Step 1  \( R_{TH} = \frac{(4+12)}{16} = 8 \ \Omega \)

Step 2  mesh analysis

\[
2 \frac{di}{dt} + 8i = 0
\]

\( i(t) = Ke^{-at}, \quad t \geq 0 \)

\( i(0^+) = 6 \ \text{A} \)

\( i(t) = 6e^{-at} \ \text{A}, \quad t \geq 0 \)
7.2 The Natural Response of an RL Circuit

Step 3: Replace L with an equivalent current source, and find $i_x$, solve the resistive circuit.

\[ i_x(t) = \frac{4 + 12}{12 + 4 + 16} \cdot i(t) \]
\[ = 3e^{-4t} \text{ A, } t \geq 0 \]

7.3 Singularity Functions

Switching functions are convenient for describing the switching actions in circuit analysis.

They serve as good approximations to the switching signals.

- unit step function $u(t)$
- unit impulse function $\delta(t)$
- unit ramp function $r(t)$
7.3 Singularity Functions

**Definition**

\[ u(t) = \begin{cases} 
0 & , \quad t < 0 \\
1 & , \quad t > 0 
\end{cases} \]

undefined at \( t = 0 \)

or more general

\[ u(t-t_0) = \begin{cases} 
0 & , \quad t < t_0 \\
1 & , \quad t > t_0 
\end{cases} \]

**Example 7**

\( V_0 u(t-t_0) \) is applied to \( a-b \) terminals

\[ V_0 \int_0^t \delta (t) \, dt = 1 \]

or \( \int_{t_0}^{t_0} \delta (t-t_0) \, dt = 1 \)

Also known as the delta function
7.3 Singularity Functions

The unit impulse function is not physically realizable but is a very useful mathematical tool.

An approximation as \( d \to 0 \)

Another approximation as \( d \to 0 \)

7.3 Singularity Functions

Shifting property, if \( t_0 \in [a, b] \)

\[
\int_{a}^{b} f(t) \delta(t-t_0) \, dt = f(t_0)
\]

\[
Q \int_{a}^{b} f(t) \delta(t-t_0) \, dt = \int_{a}^{b} f(t_0) \delta(t-t_0) \, dt = f(t_0)\int_{a}^{b} \delta(t-t_0) \, dt = f(t_0)
\]
7.3 Singularity Functions

**Definition**

\[ r(t) = \int_{-\infty}^{t} u(t)\,dt = t \, u(t) \]

or

\[
\begin{cases}
0 & , \ t \leq 0 \\
t & , \ t \geq 0
\end{cases}
\]

**Note that**

\[
\delta(t) = \frac{d}{dt} u(t), \quad u(t) = \int_{t}^{\infty} \delta(t)\,dt
\]

\[
\delta(t) = \frac{d}{dt} r(t), \quad r(t) = \int_{t}^{\infty} u(t)\,dt
\]

---

**Example 8**

\( v(t) = 10 [u(t-2) - u(t-5)] \)

\[
\frac{dv(t)}{dt} = 10\delta(t-2) - 10\delta(t-5)
\]
### 7.3 Singularity Functions

\begin{align*}
(i) \quad i(t) &= 10 u(t) - 20 u(t-2) + 10 u(t-4) \\
(d) \quad v(t) &= 5r(t) - 5r(t-2) - 10u(t-2)
\end{align*}

### 7.4 The Step Response of RC and RL Circuits

When a dc voltage (current) source is suddenly applied to a circuit, it can be modeled as a step function, and the resulting response is called step response.
7.4 The Step Response of RC and RL Circuits

Example 9

Step 1  Mesh Analysis

\[ RC \frac{dv}{dt} + v = V_s, \; t \geq 0 \]

Step 2  Solving the differential equation

(a) homogeneous solution

\[ RC \frac{dv_h}{dt} + v_h = 0 \]

\[ v_h(t) = Ke^{-\frac{t}{RC}}, \; t \geq 0 \]

(b) particular solution

\[ RC \frac{dv_p}{dt} + v_p = V_s \]

\[ v_p = V_s \]
7.4 The Step Response of RC and RL Circuits

(c) complete solution + Initial condition

\[ v(t) = v_h + v_p \]
\[ = Ke^{-\frac{t}{RC}} + V_s \]
\[ v(0^+) = v(0^-) = V_0 \]
\[ \therefore K = V_0 - V_s \]
\[ v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & t \geq 0 \end{cases} \]

\[ i(t) = C \frac{dv}{dt} = C(V_o - V_s)(\frac{-1}{RC})e^{-\frac{t}{RC}} \]
\[ = \frac{V_s - V_o}{R} e^{-\frac{t}{RC}}, \quad t \geq 0 \]

For \( V_o = 0 \), then \( i(o^+) = \frac{V_s}{R} \)

C is initially short circuited.
7.4 The Step Response of RC and RL Circuits

Example 10
Before \( t=0 \), the circuit is under steady state. At \( t=0 \), the switch is moved to B. Find \( v(t) \), \( t > 0 \)

For \( t<0 \), the circuit shown:
\[
v(0^-) = \frac{5}{3+5} \times 24V = 15V
\]

For \( t>0 \)

**Mesh analysis**
\[
4 \times 10^3 \times (0.5 \times 10^{-3}) \frac{dv}{dt} + v = 30V
\]
\[
v(t) = v_p + v_h
\]
\[
= 30 + Ke^{-\frac{t}{\tau}}, \quad t > 0
\]
\[
\tau = RC = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ second}
\]
\[
v(0^+) = v(0^-) = 15V
\]
\[
\therefore v(t) = 30 + (15 - 30)e^{-\frac{t}{\tau}}, \quad t \geq 0
\]
Example 11

Before the switch is open at \( t=0 \), the circuit is in steady state. Find \( i(t) \) & \( v(t) \) for all \( t \).

For \( t<0 \), then

\[
\begin{align*}
 v(t) &= 10V, \; t < 0 \\
 i(t) &= -\frac{10}{10}A = -1A, \; t < 0
\end{align*}
\]

For \( t>0 \), the Thevenin equivalent circuit:

\[
\begin{align*}
 v(t) &= 10V \\
 i(t) &= C \frac{dv}{dt} = \frac{1}{4} \frac{dv}{dt} \\
\frac{20}{3} \times \frac{1}{4} \frac{dv}{dt} + v &= 20V, \; t > 0 \\
v(t) &= 20 + Ke \frac{1}{3} = 20 + (10 - 20) e^{-3}, \; t > 0
\end{align*}
\]
7.4 The Step Response of RC and RL Circuits

From the original circuit

\[ KCL: i(t) = \frac{v}{20\Omega} + C \frac{dv}{dt} = 1 + e^{-0.6t}, t > 0 \]

\[ v(t) = \begin{cases} 10V, t < 0 \\ (20 - 10e^{-0.6t})V, t \geq 0 \end{cases} \]

\[ i(t) = \begin{cases} -1A, t < 0 \\ (1 + e^{-0.6t})A, t \geq 0 \end{cases} \]

Note that

\[ i(0^-) = -1A \neq i(0^+) = 2A \]

7.4 The Step Response of RC and RL Circuits

Example 12

Mesh analysis

\[ L \frac{di}{dt} + R_{TH} i = V_{TH}, \ t \geq 0 \]

\[ i(0) = 0 \]
7.4 The Step Response of RC and RL Circuits

Solution:

\[ i(t) = i_s(t) + i_p(t) \]

characteristic equation

\[ LS + R_{th} = 0 \]

\[ S = \frac{R_m}{L} = -\frac{1}{\tau} \]

\[ i_s(t) = Ke^{\frac{-t}{\tau}} \]

\[ i_p(t) = \frac{V_m}{R_{th}} \]

\[ \therefore i(t) = \frac{V_m}{R_{th}} + Ke^{\frac{-t}{\tau}} \]

\[ i(0^+) = i(0^+) = 0 \]

Note: \( L \) is equivalent to open circuit

\[ i(t) = \frac{V_m}{R_{th}}(1 - e^{\frac{-t}{\tau}}), \ t \geq 0 \]

\[ V(t) = L \frac{di}{dt} = V_m e^{\frac{-t}{\tau}} u(t) \]

\[ i(t) = \frac{V_m}{R_{th}}(1 - e^{\frac{-t}{\tau}}), \ t \geq 0 \]

\[ V(t) = L \frac{di}{dt} = V_m e^{\frac{-t}{\tau}} u(t) \]
7.4 The Step Response of RC and RL Circuits

Example 13

For $t<0$

$$i(0^-) = \frac{10}{2} = 5A$$

For $t=0$, switch is opened

$$t<0, \text{ steady state}$$

Mesh Analysis

$$\frac{1}{3} \frac{di}{dt} + 5i = 10V, \ t \geq 0$$

$$s = -15$$

$$i_n(t) = Ke^{-t^\tau}, \ \tau = \frac{1}{15}$$

$$i_p(t) = 2A$$

$$i(t) = i_n + i_p$$
7.4 The Step Response of RC and RL Circuits

initial condition

\[ i(0^+) = i(0^-) = 5A \]

\[
\begin{align*}
\therefore i(t) &= 2 + 3e^{\frac{-t}{\tau}}, \quad t \geq 0
\end{align*}
\]

Note that if \( V \) is chosen for solution:

\[ V(0^-) = 0V \]

KVL: \( V(0^+) = 10V - (2 + 3)5 = -15V \)

or \[ V = L \frac{di}{dt} = \frac{1}{3} \frac{d}{dt} (2 + 3e^{\frac{-t}{\tau}}) \]

\[ = -\frac{1}{\tau} e^{\frac{-t}{\tau}}, \quad t \geq 0^+ \]

\[ V(0^+) = -15V, \quad same \ answer. \]
7.4 The Step Response of RC and RL Circuits

Example 14: $S_1$ is closed at $t=0$  
Find $i(2)=? \quad S_2$ is closed at $t=4$  
$i(5)=?$

For $t<0$  
$t<0, i=0$, open circuit
7.4 The Step Response of RC and RL Circuits

For $0^+ \leq t < 4S$

\[ 5 \frac{di}{dt} + 10i = 40V, \quad 0^+ \leq t \leq 4S \]

\[ i(0^+) = i(0^-) = 0A \]

\[ \tau = \frac{L}{R} = \frac{5}{10} = \frac{1}{2} \]

\[ \therefore i(t) = 4(1 - e^{-2t})A, \quad 0^+ \leq t \leq 4S \]

\[ i(4^-) = 4(1 - e^{-8}) = 4A \]
7.4 The Step Response of RC and RL Circuits

For \( t \geq 4^+ \)

\[
i(4^-) = 4A
\]

\[
V_{TH} = \frac{40 - 10}{4 + 2} \times 2 + 10 = 20V
\]

\[
R_{TH} = 4\Omega \parallel 2\Omega + 6\Omega = \frac{4}{3} + 6 = \frac{22}{3} \Omega
\]

\[
\tau = \frac{L}{R_{TH}} = \frac{5}{22/3} = \frac{15}{22} \text{ s}
\]
7.4 The Step Response of RC and RL Circuits

For $t \geq 4^+$

$$i(t) = \frac{30}{11} + Ke^{\frac{22}{15} (t-4)}, \ t \geq 4^+$$

$$i(t) = \frac{30}{11} + K$$

$$K = 4 - \frac{30}{11} = \frac{14}{11}$$

$$\therefore i(t) = \frac{30}{11} + \frac{14}{11} e^{\frac{22}{15} (t-4)}, \ t \geq 4^+$$

---

**In Summary**

$$i(t) = \begin{cases} 
0, & t \leq 0 \\
4(1 - e^{-2t}), & 0 \leq t \leq 4 \\
\frac{30}{11} + \frac{14}{11} e^{-22(t-4)}, & t \geq 4 
\end{cases}$$
Summary

Objective 1: Be able to determine the natural response of both RC and RL circuits.

Objective 2: Be able to find the step response of both RC and RL circuits.

Objective 3: Know and be able to use the singularity functions.

Objective 4: Be able to analyze circuits with sequential switching.

Chapter Problems: 7.7
7.14
7.23
7.32
7.47
7.63

Due within one week.