CHAPTER 6

CAPACITANCE, INDUCTANCE, AND MUTUAL INDUCTANCE

CONTENTS

6.1 The Capacitor
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6.3 Series-Parallel Combinations of Capacitance and Inductance
6.4 Mutual Inductance
In this chapter, two new and important passive linear components are introduced. They are ideal models. Resistors dissipate energy but capacitors and inductors are energy storage components.

### 6.1 The Capacitor

*Circuit symbol and component model.*

\[ q(t) = C \int v_c(\tau) \, d\tau \]

- \( q \): charge
- \( C \): capacitance, in \( F \) (Farad)
6.1 The Capacitor

\[ v_c(t) = \frac{1}{C} \int_{\tau_0}^{t} i_c(\tau) \, d\tau = \frac{1}{C} \int_{-\infty}^{\tau_0} i_c(\tau) \, d\tau + \frac{1}{C} \int_{\tau_0}^{t} i_c(\tau) \, d\tau \]

\[ v_c(t) @ v_c(t_0) + \frac{1}{C} \int_{\tau_0}^{t} i_c(\tau) \, d\tau \quad \text{......... (A)} \]

\[ i_c(t) = \frac{dq}{dt} = C \frac{dv_c}{dt} \quad \text{................... (B)} \]

1 farad = 1 Coulomb/Volt

The unit of capacitance is chosen to be farad in honor of the English physicist, Michael Faraday (1791–1867).

Example 1: A parallel-plate capacitor

\[ C = \varepsilon \frac{A}{d} \]

\( \varepsilon \): the permittivity of the dielectric material between the plates
6.1 The Capacitor

Example 1: (cont.)

If \( v_c > 0 \) and \( i_c > 0 \), or \( v_c < 0 \) and \( i_c < 0 \) the capacitor is being charged.
If \( v_c \cdot i_c < 0 \), the capacitor is discharging.

\[
p(t) = v_c(t) i_c(t) = v_c(t) \frac{dv_c}{dt}
\]

Energy in a capacitor

\[
w = \int_{-\infty}^{\infty} p(\tau) d\tau = \frac{1}{2} C v_c^2(t) = \frac{q^2}{2C} \geq 0
\]

6.1 The Capacitor

(a) When \( v_c \) is constant, then \( i_c = 0 \).
    i.e. , equivalent to open circuit

(b) \( v_c(t) \) is a continuous function
    if there is only finite strength energy sources inside the circuit.

\[
Q v_c(t + \epsilon) = v_c(t) + \frac{1}{C} \int_{t}^{t+\epsilon} i_c(\tau) d\tau
\]

\[
\lim_{\epsilon \to 0} v_c(t + \epsilon) = v_c(t) , \quad v_c(0^+) = v_c(0^-)
\]
i.e. \( v_c(t) \) can not change abruptly for finite \( i_c(t) \)
6.1 The Capacitor

(c) An ideal capacitor does not dissipate energy. It stores energy in the electrical field.

(d) A nonideal capacitor has a leakage resistance

\[ R_c \]

ESR: equivalent series resistance

6.2 The Inductor

Circuit symbol and component model.

\[ \lambda = \int_0^t v_L(\tau) d\tau \]

\( \lambda \) : flux linkage, in weber-turns

\( L \) : inductance, in H (Henry)
6.2 The Inductor

\[ i_L = \frac{1}{L} \int_{-\infty}^{\tau} v_L(\tau) \, d\tau = \frac{1}{L} \int_{-\infty}^{t_0} v_L(\tau) \, d\tau + \frac{1}{L} \int_{t_0}^{\tau} v_L(\tau) \, d\tau \]

\[ i_L(t) = i_L(t_0) + \frac{1}{L} \int_{t_0}^{t} v_L(\tau) \, d\tau \quad \text{---------------(A)} \]

\[ v_L(t) = \frac{d\lambda}{dt} = L \frac{di_L(t)}{dt} \quad \text{---------------(B)} \]

\[ p(t) = v_L i_L = L i_L(t) \frac{di_L(t)}{dt} \]

Energy in an inductor

\[ w(t) = \int p(\tau) \, d\tau = \frac{1}{2} Li_L^2(t) \geq 0 \]

---

Example 2: An Inductor

\[ \int H \, dl = Ni \]

\[ Hl = Ni, \quad H = \frac{Ni}{l} \]

\[ B = \mu H = \frac{\mu Ni}{l}, \quad \mu \text{ permeability of the core} \]

H : magnetic field intensity
B : flux density
\[ \phi : \int B \, dA \quad \text{flux} \]
6.2 The Inductor

Example 2: (cont.)

\[ \phi = BA = \frac{\mu ANi}{l} \]
\[ \lambda = N\phi = \frac{\mu AN^2i}{l} \text{, flux linkage} \]
\[ \therefore L = \frac{\lambda}{i} = \frac{\mu AN^2}{l} \]

6.2 The Inductor

The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797-1878).

\[ 1 \text{H} = 1 \text{volt-second / ampere} \]

(a) when \( i_L \) is constant, then \( v_L = 0 \).

\( \text{i.e.} \), equivalent to short circuit

(b) \( i_L(t) \) is a continuous function if there is only finite strength source inside the circuit.

\[ i_L(t+\varepsilon) = i_L(t) + \frac{1}{L} \int_{t-\varepsilon}^{t+\varepsilon} v_L(\tau) \, d\tau \]

\[ \lim_{\varepsilon \to 0} i_L(t+\varepsilon) = i_L(t), \quad \text{or} \quad i_L(0^+) = i_L(0^-) \]

\( \text{i.e.} \, i_L(t) \) can not change abruptly for finite \( v_L(t) \).
6.2 The Inductor

(c) An ideal inductor does not dissipate energy. It stores energy in the magnetic field.

(d) A nonideal inductor contains winding resistance and parasitic capacitance.

Example 3: Under dc and steady state conditions, find (a) $I$, $V_C$ & $I_L$, (b) $W_C$ and $W_L$.

\[ I = I_L = \frac{12}{1+5} = 2A \]
\[ V_C = 5I_L = 10V \]
\[ W_C = \frac{1}{2} \times 1 \times 10^2 = 50J \]
\[ W_L = \frac{1}{2} \times 2 \times 2^2 = 4J \]
6.2 The Inductor

(a) The capacity of C and L to store energy makes them useful as temporary voltage or current sources, i.e., they can be used for generating a large amount of voltage or current for a short period of time.

(b) The continuity property of \( V_C(t) \) and \( i_L(t) \) makes inductors useful for spark or arc suppression and for converting pulsating voltage into relatively smooth dc voltage.

(c) The frequency sensitive property of L and C makes them useful for frequency discrimination.

(eg. LP, HP, BP filters)
6.3 Series-Parallel Combinations of Capacitance and Inductance

N capacitors in parallel

\[ i = i_1 + i_2 + \cdots + i_N \]

\[ Q_i = \frac{dv}{dt} + \frac{dv}{dt} + \cdots + \frac{dv}{dt} \]

\[ = \sum_{k=1}^{N} c_k \frac{dv}{dt} = \frac{dv}{dt} \]

\[ \therefore C_{eq} = c_1 + c_2 + \cdots + c_N \]

\[ v(0) = v_i(0) \]

N capacitors in series

\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \]

\[ v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0) \]
6.3 Series-Parallel Combinations of Capacitance and Inductance

N inductors in series

\[ i = \frac{L_1}{v} + v_1 - + v_2 - + \cdots + v_N - \]

\[ Qv = v_1 + v_2 + L + v_N \]

\[ = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L + L_N \frac{di}{dt} \]

\[ = \left( \sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt} \]

\[ \therefore L_{eq} = L_1 + L_2 + L + L_N \]

\[ i(0) = i_1(0) = i_2(0) = \cdots = i_N(0) \]

\[ i_1(t) = i_2(t) = \cdots = i_N(t) = i(t) \]

\[ i(0) = i_1(0) = i_2(0) = \cdots = i_N(0) \]

6.3 Series-Parallel Combinations of Capacitance and Inductance

N inductors in parallel

\[ i = \frac{L_1}{v} + \frac{L_2}{v} + \cdots + \frac{L_N}{v} \]

\[ QV_1 = V_2 = L = V_N = V \]

\[ \therefore i = i_1 + i_2 + L + i_N \]

\[ i(t_0) = i_1(t_0) + i_2(t_0) + L + i_N(t_0) \]

\[ \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \]

\[ = \frac{1}{L_{eq}} \int uL + \frac{1}{L_{eq}} \int i(t_0) \]

\[ = \frac{1}{L_{eq}} \int uL + \frac{1}{L_{eq}} \int i(t_0) \]

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6.3 Series-Parallel Combinations of Capacitance and Inductance

<table>
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<th>Resistor</th>
<th>Capacitor</th>
<th>Inductor</th>
</tr>
</thead>
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<tr>
<td>V-I</td>
<td>$V = RI$</td>
<td>$v = v(t_0) + \frac{1}{C} \int_0^t i , dt$</td>
<td>$v = L \frac{di}{dt}$</td>
</tr>
<tr>
<td>I-V</td>
<td>$I = \frac{1}{R} V$</td>
<td>$i = C \frac{dv}{dt}$</td>
<td>$i = i(t_0) + \frac{1}{L} \int_{t_0}^t v , dt$</td>
</tr>
<tr>
<td>P or W</td>
<td>$P = \frac{V^2}{R} = I^2 R$</td>
<td>$W = \frac{1}{2} C v^2$</td>
<td>$W = \frac{1}{2} L i^2$</td>
</tr>
<tr>
<td>series</td>
<td>$R_{eq} = R_1 + R_2$</td>
<td>$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$</td>
<td>$L_{eq} = L_1 + L_2$</td>
</tr>
<tr>
<td>parallel</td>
<td>$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$</td>
<td>$C_{eq} = C_1 + C_2$</td>
<td>$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$</td>
</tr>
<tr>
<td>dc case</td>
<td>same</td>
<td>open circuit</td>
<td>short circuit</td>
</tr>
</tbody>
</table>

In summary:

6.4 Mutual Inductance

Circuit symbol and model of coupling inductors

$L_1$, $L_2$ : self inductances
$M$ : mutual inductance
unit : in H

$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$

$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$
6.4 Mutual Inductance

Example 4: Mutual inductance

Apply $I_1$, with $i_2=0$

$$\Phi = \oint H \cdot dl = N_1 \cdot I_1$$

6.4 Mutual Inductance

Assume uniform magnetic field intensity $H$

$$H = \frac{N_1 I_1}{l}$$

$$\therefore B = \mu H = \frac{\mu N_1 I_1}{l}$$

$$\Phi = \int B \cdot dA = \frac{\mu A N_1^2 I_1}{l}$$

$$\lambda_1 = N_1 \Phi = \frac{\mu A N_1^2 I_1}{l}$$

$$\lambda_2 = N_2 \Phi = \frac{\mu A N_1 N_2 I_1}{l}$$

$$L_1 \frac{\lambda_1}{I_1} = \frac{\mu A N_1^2}{l}$$

$$M \frac{\lambda_2}{I_1} = \frac{\mu A N_1 N_2}{l}$$
6.4 Mutual Inductance

Dot convention for mutually coupled inductors:
When the reference direction for a current enters the dotted terminal of a coil, the polarity of the induced voltage in the other coil is positive at its corresponding dotted terminal.

Another dot of coil 2 should be placed in terminal c.
Conceptually, one can connect a resistor at cd terminals. Then $i_2$ will be negative. The generated flux of $i_2$ will oppose the increasing of $\phi$ due to increasing $i_1$ (Lentz law). Hence, another dot should be placed at c terminal.

In case, the other dot is placed at d terminal, then $i_2$ will be positive. Hence, the generated flux of $i_2$ will be added to the increasing $\phi$ due to $i_1$. 
6.4 Mutual Inductance

Then the induced voltage at coil two will increase and so will $i_2$. This will violate the conservation of energy.

The procedure for determining dot markings

**Step 1** Assign current direction references for the coils.

**Step 2** Arbitrarily select one terminal of one coil and mark it with a dot.

**Step 3** Use the right-hand rule to determine the direction of the magnetic flux due to the current of the other coil.
6.4 Mutual Inductance

Step 4: If this flux direction has the same direction as that of the first dot terminal current, then the second dot is placed at the terminal where the second current enters. Otherwise, the second dot should be placed at the terminal where the second current leaves.

Example 5: Determining dot markings

Step 1: Assign $i_1$, $i_2$, $i_3$ directions.
6.4 Mutual Inductance

Example 5: (cont.)

Step 2:

For coils 1 and 2, choose first dot as follows

For coils 1 and 3

C.T. Pan
6.4 Mutual Inductance

Example 5: (cont.)

For coils 2 and 3

Step 3:
Check the relative flux directions and determine the dot position at the other coil.

For coil 1 and 2
\( \phi_1 \) and \( \phi_2 \) are in the same direction.
6.4 Mutual Inductance

Example 5: (cont.)

For coil 1 and 3

\( \Phi_1 \) and \( \Phi_3 \) are in opposite direction

For coil 2 and 3

\( \Phi_2 \) and \( \Phi_3 \) are in opposite direction
6.4 Mutual Inductance

Example 5: (cont.)

In summary

\[
\begin{align*}
v_1 &= L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} - M_{13} \frac{di_3}{dt} \\
v_2 &= M_{12} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} - M_{23} \frac{di_3}{dt} \\
v_3 &= -M_{13} \frac{di_1}{dt} - M_{23} \frac{di_2}{dt} + L_3 \frac{di_3}{dt}
\end{align*}
\]

If the physical arrangement of the coils are not known, the relative polarities of the magnetically coupled coils can be determined experimentally. We need

- a dc voltage source \( V_S \)
- a resistor \( R \) to limit the current
- a switch \( S \)
- a dc voltmeter
6.4 Mutual Inductance

Step 1: Assign current directions and arbitrarily assign one dot at coil one.

\[ \begin{align*}
&\text{ } + \hspace{1cm} i_1 \hspace{1cm} + \\
&\text{ } \hspace{1cm} v_1 \hspace{1cm} L_1 \\
&\hspace{1cm} L_2 \hspace{1cm} v_2 \\
&\text{ } - \hspace{1cm} \text{ } \hspace{1cm} -
\end{align*} \]

Step 2: Connect the setup as follows.

\[ \begin{align*}
&\text{ } + \hspace{1cm} i_1 \hspace{1cm} + \\
&\text{ } \hspace{1cm} v_1 \hspace{1cm} L_1 \\
&\hspace{1cm} L_2 \hspace{1cm} v_2 \\
&\text{ } - \hspace{1cm} \text{ } \hspace{1cm} -
\end{align*} \]
6.4 Mutual Inductance

Step 3: Determine the voltmeter deflection when the switch is closed.
If the momentary deflection is upscale, then

\[ i_1 = 0, \quad i_2 \text{ increased from zero to } I. \]

If the momentary deflection is downscale, then

\[ \text{The reciprocal property of the mutual inductance can be proved by considering the energy relationship.} \]

Step 1: \( i_2 = 0, \ i_1 \text{ increased from zero to } I. \)
**6.4 Mutual Inductance**

Step 2: keep $i_1 = I_1$, $i_2$ is increased from zero to $I_2$

\[
\begin{align*}
  v_1 &= L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} = M_{12} \frac{di_2}{dt} \\
  v_2 &= M_{21} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = L_2 \frac{di_2}{dt}
\end{align*}
\]

Input power

\[
p_2 = v_1 i_1 + v_2 i_2 = M_{12} i_1 \frac{di_2}{dt} + (L_2 \frac{di_2}{dt}) i_2
\]

Input energy

\[
w_2 = \int p_2 \, dt = \int_0^{i_2} M_{12} i_1 \, di_2 + \int_0^{i_2} L_2 i_2 \, di_2
\]
\[
= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2
\]

Total energy when $i_1 = I_1$, $i_2 = I_2$

\[
w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + I_1 I_2 M_{12}
\]

Similarly, if we reverse the procedure, by first increasing $i_2$ from zero to $I_2$ and then increasing $i_1$ from zero to $I_1$, the total energy is

\[
w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + I_1 I_2 M_{21}
\]

Hence, $M_{12} = M_{21} = M$
6.4 Mutual Inductance

Definition of coefficient of coupling $k$

$$k \leq \frac{M}{\sqrt{L_1 L_2}}$$

$0 \leq k \leq 1$

$0 < k < \frac{1}{2}$, loosely coupling

$\frac{1}{2} \leq k < 1$, closely coupling

$k = 1$, unity coupling

According to dot convention chosen, the total energy stored in the coupled inductors should be

$$w = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2 \geq 0$$

In particular, consider the limiting case

$$\frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 - M i_1 i_2 = 0$$

The above equation can be put into the following form

$$\left(\sqrt{\frac{L_1}{2}} i_1 - \sqrt{\frac{L_2}{2}} i_2\right)^2 + i_1 i_2 \left(\sqrt{L_1 L_2} - M\right) = 0$$
6.4 Mutual Inductance

Thus, \( w(t) \geq 0 \) only if

\[
\sqrt{L_1 L_2} \geq M
\]

when \( i_1 \) and \( i_2 \) are either both positive or both negative

Hence, \( k \leq 1 \)

Example 6: Finding mesh-current equations for a circuit with magnetically coupled coils.

Three meshes and one current source, only need two unknowns, say \( i_1 \) and \( i_2 \).

Note: current \( i_{4H} = i_1(t) \)

current \( i_{16H} = i_1 - i_2 \)
Due to existence of mutual inductance $M=8\,\text{H}$, there are two voltage terms for each coil. One can use dependence source to eliminate the coupling relation as follows.

\[
\begin{align*}
4\frac{di_1}{dt} + 8\frac{d}{dt}(i_g - i_2) + 20(i_1 - i_2) + 5(i_1 - i_g) &= 0 \\
\text{For } i_2 \text{ mesh} \\
20(i_2 - i_1) + 60i_2 + 16\frac{d}{dt}(i_2 - i_g) - 8\frac{d}{dt}i_1 &= 0
\end{align*}
\]
Summary

- Objective 1: Know and be able to use the component model of an inductor.
- Objective 2: Know and be able to use the component model of a capacitor.
- Objective 3: Be able to find the equivalent inductor (capacitor) together with its equivalent initial condition for inductors (capacitors) connected in series and in parallel.

Chapter Problems:

6.4
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Due within one week.