CH3 Wave Properties of Particles

De Broglie waves

A moving body behaves in certain ways as though it has a wave nature.

* for photon

\[ P = \frac{h \nu}{c} = \frac{h}{\lambda} \]

Photon wavelength \( \lambda = h/P \) \((3.1)\)

De Broglie suggested (3.1) is a general one that applies to material particles as well as to photons.

De Broglie wavelength

\[ \lambda = h/P = h/mv \]

\[(m = \frac{m_o}{\sqrt{1 - v^2/c^2}})\]
Example 3.1

Find the de Brogli wavelengths of

(a) 46-g golf ball with a $v = 30$ m/s

(b) e’ with a $v = 10^7$ m/s

1. $v \ll c \rightarrow m = m_o$

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{Js}}{(0.046 \text{kg})(30 \text{m/s})} = 4.8 \times 10^{-34} \text{m}$$

wavelength is very small

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{Js}}{(9.1 \times 10^{-31} \text{kg})(10^7 \text{m/s})} = 7.3 \times 10^{-11} \text{m}$$

$= 0.73 \text{Å}$

the radius of H atom $= 5.3 \times 10^{-11} \text{m} = 0.53 \text{ Å}$

wave character of moving e’ is the key to understand atomic structure behavior
3.2 Waves of probability

Water wave \( \longrightarrow \) (varing quantity) height of water surface

Light wave \( \longrightarrow \) E& H fields

How about matter waves

\( \longrightarrow \) Wave function \( \Psi \)

The value of wave function associated with a moving body at the particular point \( x, y, z \) at time \( t \) is related to the likelihood of finding the body there at the time.

* \( \Psi \) has no direct physical significance

\[ 0 \leq \text{probability} \leq 1 \]

but the amplitude of wave am be positive or negative

\( \longrightarrow \) no negative probability

\( \longrightarrow \) \( |\phi|^2 \) : square of the absolute value of wave function

probability density

** The probability of experimentally finding the body described by the wave function \( \Psi \) at the point \( x, y, z \) at time \( t \) is proportional to \( |\phi|^2 \) there at \( t \).

wave function \( \Psi \) that described a particle is spread out is spall, but it does not mean that the particle itself is spread out.
3.3 Describing a wave

de Broglie wave velocity $v_p$

$$v_p = \nu \lambda \quad (\lambda = h/mv)$$

$$h \nu = mc^2 \rightarrow \nu = mc^2/h$$

De Broglie phase velocity $v_p = \nu \lambda = (mc^2/h)(h/mv) = c^2/v \quad (v = \text{particle velocity})$

Because $V < C$

$\rightarrow$ de Broglie waves always travel faster than light !!

$\rightarrow$ Phase velocity, group velocity.

Figure 3.1 (a) The appearance of a wave in a stretched string at a certain time. (b) How the displacement of a point on the string varies with time.
At $x=0$, $y=A\cos(2\pi vt)$ for time $t$

$x = v_p t$, $t = x/v_p$

$y = A\cos(2\pi v(t - x/v_p))$

the amplitude for $y(x,t) = y(0,t-x/v_p)$

$y = A\cos(2\pi(v t - \frac{x}{v_p}))$  $v_p = \nu \lambda$

$\rightarrow y = A\cos(2\pi(v t - x/\lambda))$

angular frequency $\omega = 2\pi$  $\nu$  wave number $k = 2\pi / \lambda = \omega / v_p$

$\rightarrow y = A\cos(\omega t - kx)$

Figure 3.2 Wave propagation.

Figure 3.3 A wave group.

Figure 3.4 Beats are produced by the superposition of two waves with different frequencies.
The amplitude of de Broglie waves → probability

De Broglie wave can not be represented by \( y = A \cos(\omega t - kx) \)

. wave representation of a moving body → wave packet

. wave group

. An example is a beat. (two sound waves of the same amplitude but slightly different frequencies)

original 440, 442 Hz → hear fluctuating sound of 441 Hz with 2 beats/s

a wave group: superposition of individual waves of different \( \lambda \)

which interference with one another

→ variation in amplitude → define the group shape
(1) If the velocities of the waves are the same → the velocity of wave group is common phase velocity

(1) If the phase velocity varies with \( \lambda \) → an effect called dispersion

→ individual waves do not proceed together

→ wave group has a velocity different from the phase velocities

→ the case of de Broglie wave

- **group velocity**

\[
y_1 = A\cos[(\omega t - ks)]
\]

\[
y_2 = A\cos[(\omega + \Delta \omega) t - (k + \Delta k)x]
\]

→ \( y = y_1 + y_2 = 2A\cos\frac{1}{2}[(2\omega + \Delta \omega)t - (2k + \Delta k)x] \cos\frac{1}{2}(\Delta \omega t - \Delta kx) \)

because \( \Delta \omega \ll \omega \) → \( 2\omega + \Delta \omega \approx 2\omega \)

\( \Delta k \ll k \) → \( 2k + \Delta k \approx 2k \)

→ \( Y = 2A\cos(\omega t - kx)\cos[(\Delta \omega /2)t - (\Delta k/2)x] \)

A wave of angular frequency \( \omega \) & wave number \( k \) that has superimposed upon it a modulation of angular frequency \( 1/2 \Delta \omega \) & of wave number \( 1/2 \Delta k \)
Modulation produce wave group

\[ v_p = \frac{\omega}{k} = \frac{2\pi v}{\frac{2\pi}{\lambda}} = \nu \lambda \quad \text{phase velocity} \]

\[ v_g = \Delta \omega / \Delta k = d\omega / dk \quad \text{group velocity} \]

for de Broglie waves

\[ \sigma = 2\pi \nu = \frac{2\pi \nu mc^2}{\hbar} = \frac{2\pi m \nu c^2}{\hbar \sqrt{1 - \nu^2 / c^2}} \quad \text{(because } \hbar \nu = mc^2) \]

\[ k = \frac{2\pi}{\lambda} = \frac{2\pi n \nu}{\hbar} = \frac{2\pi m \nu}{\hbar \sqrt{1 - \nu^2 / c^2}} \quad \text{(because } \lambda = \hbar / mv) \]

* both \( \omega \) &\( k \) are functions of body’s \( v \) \quad \nu = d\omega / dk = \frac{d\omega}{dv} \]

\[ \frac{d\omega}{dv} = \frac{2\pi m \nu}{\hbar \left(1 - \nu^2 / c^2\right)^{\frac{3}{2}}} \quad , \quad \frac{dk}{dv} = \frac{2\pi m \nu}{\hbar \left(1 - \nu^2 / c^2\right)^{\frac{3}{2}}} \]

\[ \longrightarrow v_g = v \quad \text{(de Broglie group velocity)} \]

De Broglie wave group associated with a moving body travels with the same velocity as the body.

De Broglie phase velocity \( v_p = \omega / k = c^2 / \nu \)

\( v_p > \) velocity of the body \( v > c \)

(\( \therefore \) it is not the motion of the body)
Ex 3.3:

An e' has a de Broglie wavelength of $2\text{pm}=2\times10^{-12}\text{m}$. Find its kinetic energy & the phase & group velocity of its de Broglie waves.

(a) $E = E_o + kE \quad \rightarrow \quad kE = E - E_o = \sqrt{E_o^2 + p^2c^2} - E_o$

$$pc = \frac{hc}{\lambda} = \frac{(4.136\times10^{-15}\text{ev.s})(3\times10^8\text{m/s})}{(2\times10^{-12})} = 6.2\times10^5\text{ev}=620\text{kv}$$

the rest energy of e' is $E_o = 511\text{kv}$

$$kE = \sqrt{(511)^2 + (620)^2} - 511 = 292\text{ kev}$$

(b) e' velocity

$$E = \frac{E_o}{\sqrt{1-v^2/c^2}} \quad \rightarrow \quad v = c\sqrt{1-\frac{E_o^2}{E^2}} = 0.771c$$

$\therefore$ $v_p = c^2/v = 1.3c$, $v_g = v = 0.771c$
3.5 particle diffraction $\rightarrow$ e'-beam diffraction $\rightarrow$ confirm de Broglie waves

The method of plotting is such that the intensity at any angle is proportional to the distance of the curve at the angle from the point of scattering.

$$n \lambda = 2 \sin \theta \quad \rightarrow \quad \lambda = 2 \sin \theta = 0.165 \text{nm} \quad \lambda = \frac{h}{mv} = 0.166 \text{nm}$$
Figure 3.8 Because the wavelengths of the fast electrons in an electron microscope are shorter than those of the light waves in an optical microscope, the electron microscope can produce sharp images at higher magnifications. The electron beam in an electron microscope is focused by magnetic fields.

Figure 3.9 A particle confined to a box of width $L$. 

Electron source

Magnetic condensing lens

Object

Magnetic objective lens

Electron paths

Magnetic projection lens

Image

L
3.6 particle in a box

A particle trapped in a box = a standing wave. \( \Psi \) must be zero at the walls

\[ \lambda_n = \frac{2L}{n} \quad n=1,2,3 \ldots \]

De Broglie wavelength of trapped particles.

\[ KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{\lambda^2} \]

\[ \therefore \lambda_n = \frac{2L}{n} \quad KE + v = E_n \]

The energy for the particle in a box

\[ E_n = n^2 \frac{h^2}{8mL^2} \quad n=1,2,3 \ldots \]

Each permitted energy is called an energy level. \( n=\text{quantum number} \)

This can be applied to any particle confined to a certain region of space.

\[ \therefore \text{For example} \]

\[ e^- \]
1. A trapped particle cannot have an arbitrary energy, as a free particle can.

Confinement leads to restriction on its wave function that alloy the particle to have certain energies.

2. A trapped particle cannot have zero energy.

\[ \therefore \quad \text{de Broglie wavelength } \lambda = \frac{h}{mv} \quad \text{If } v = 0 \rightarrow \lambda = \infty \]

\[ \rightarrow \quad \text{it can not be a trapped particle.} \]

3. \[ \therefore \quad h = 6.63 \times 10^{-34} \text{Js} \quad \text{very small} \]

\[ \therefore \quad \text{only if } m \& L \text{ are very small, or we are not aware of energy quantization in our own experience.} \]

Ex 3.4

An e' is in a box 0.1nm across, which is the order of magnitude of atomic distance, find its permitted energy.

\[ m = 9.1 \times 10^{-31} \text{kg} \quad \& \quad L = 0.1 \text{nm} = 10^{-10} \text{m} \]

\[ E_n = n^2 \left( \frac{6.63 \times 10^{-34}}{8 \times (9.1 \times 10^{-31}) (10^{-10})^2} \right) = 6 \times 10^{-18} n^2 \text{J} = 38n^2 \text{ev} \]

When \( n = 1 \rightarrow 38 \text{ ev} \)

\[ n = 2 \rightarrow 152 \text{ev} \quad \text{see fig 3.11} \]

\[ n = 3 \rightarrow 342 \text{ ev} \]
Ex 3.5

A long marble is in a box 10 cm across, find its permitted energies

$$E_n = 5.5 \times 10^{-64} n^2 J \quad n=1$$

$$E = 5.5 \times 10^{-64} J \quad \rightarrow \quad v = 3.3 \times 10^{-31} \text{ m/s}$$

Which can not be experimentally distinguished from a stationary marble.

For a reasonable speed 1/3 m/s \( \rightarrow \quad n = 10^{30}!! \)

Energy levels are very close \( \rightarrow \) quantum effects are imperceptible

- Uncertainty principle

  * wave group narrower \( \rightarrow \) particles position precise.

  However, \( \lambda \) of waves in a narrow packet is not well defined \( \therefore \lambda = h/mv \quad \therefore \) \( P \) is not precise

  * A wide wave group \( \rightarrow \) clearly defined \( \lambda \)

  but position is not certain
uncertainty principle:

It is impossible to know both the exact position & exact momentum of an object at the same time.

An infinite # of wave trains with different frequencies wave numbers and amplitude is required for an isolated group of arbitrary shape.

$$\varphi(x) = \int_{0}^{\infty} g(k) \cos(kx) dk$$  Fourier integral

g(k): amplitude of the waves varying with k, furrier transform of \( \varphi(x) \)

Figure 3.13 An isolated wave group is the result of superposing an infinite number of waves with different wavelengths. The narrower the wave group, the greater the range of wavelengths involved. A narrow de Broglie wave group thus means a well-defined position (\( \Delta x \) smaller) but a poorly defined wavelength and a large uncertainty \( \Delta p \) in the momentum of the particle the group represents. A wide wave group means a more precise momentum but a less precise position.

Figure 3.14 The wave functions and Fourier transforms for (a) a pulse, (b) a wave group, (c) an wave train, and (d) a gaussian distribution. A brief disturbance needs a broader range of frequencies to describe it than a disturbance of greater duration. The Fourier transform of a gaussian function is also a gaussian function.
wave numbers needed to represent a wave group extend from \( k=0 \) to \( k=\infty \), but for a group which length \( \Delta x \) is finite waves which amplitudes \( g(k) \) are appreciable have wave numbers that lie within a finite interval \( \Delta k \). The shorter the group, the broader the range of wave numbers needed.

*Gaussian function: \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \)

Standard deviation \( \sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - x_0)^2} \) (square-root-mean)

Width of a gaussian curve at half its max is 2.35 \( \sigma \)

\[ p_{x_0 \pm \sigma} = \int_{x_0 - \sigma}^{x_0 + \sigma} f(x) \, dx = 0.683 \]
Min $\Delta x \Delta k$ occur for Gaussian function

Take $\Delta x, \Delta k$ as standard deviation of $\varphi(x) \& g(x)$ $\Rightarrow \Delta x \Delta k = 1/2$

$\therefore$ in general $\Delta x \Delta k \geq 1/2$

$\therefore k = 2\pi / \lambda = 2\pi P/h \Rightarrow P = h\Delta k/2\pi \Rightarrow \Delta P = h\Delta k/2\pi$

$\therefore \Delta x \Delta k \geq 1/2 \quad \Delta k \geq 1/2 \Delta x$

$\Rightarrow \Delta x \Delta p \geq h/4\pi \quad (\therefore \Delta x(h\Delta k/2\pi) \geq h/4\pi)$

$\Rightarrow \Delta x \Delta p \geq \frac{h}{2} \quad [h = h/2\pi]$

Ex 3.6

A measurement establishes the position of a proton with an accuracy of $\pm 1.00 \times 10^{-11}$ m. Find the uncertainty in the proton’s position 1.00 s later. Assume $v \ll c$

Sol: At time $t=0$, uncertainty in position $\Delta x_0 = 1.00 \times 10^{-11}$ m

$\Rightarrow$ The uncertainty in $P$ at this time $\geq \frac{h}{2\Delta x_0}$

$\therefore \Delta P = m_0 \Delta v \Rightarrow \Delta v = \Delta P/m_0 \geq \frac{h}{2m_0 \Delta x_0}$

$\Delta x = t \Delta v \geq \frac{ht}{2m_0 \Delta x_0} = 3.15 \times 10^3$ m $\quad (\therefore \Delta x \propto 1/\Delta x_0)$

*the more we know at $t=0$, the less we know at $t=t*$
look at e' light of wavelength $\lambda \rightarrow P=\frac{h}{\lambda} \rightarrow$ when one of three photons bounces off the e' e' momentum is changed. The exact P cannot be predicted, but $\Delta P \sim \frac{h}{\lambda}$ (the order of magnitude as P) $\Delta x \sim \lambda$

ie if we use shorter $\lambda \rightarrow$ increase accuracy of position

$\rightarrow$ higher photon momentum disturb e' motion more

$\rightarrow$ accuracy of the momentum measurement deceasing

$\rightarrow \Delta x \Delta P \geq h$ (consist with $\Delta x \Delta P \geq \frac{h}{2}$)

(1) If the energy is in the form of em waves, the limited time available restricts the accuracy with which we can determine the frequency $\nu$.

(2) Assume the min uncertainty in the number of waves we count in a wave group is one wave.
\[\therefore \text{Frequency of wave} = \text{# of wave/ time interval} \rightarrow \Delta \nu \geq \frac{1}{\Delta t}\]

\[\therefore E = h \Delta \nu \rightarrow \Delta E \geq \frac{h}{\Delta t} \quad \text{or} \quad \Delta E \Delta t \geq h\]

more precise calculation \[\rightarrow \Delta E \Delta t \geq \frac{h}{2}\]

**ex 3.9**

An “excited” atom gives up its excess energy by emitting a photon of characteristic frequency. The average period that elapses between the excitation of an atom & the time it radiates is \(1.0 \times 10^{-9}\) s. find the uncertainty in the frequency of the photon.

\[\Delta E \geq \frac{h}{2} \Delta t = 5.3 \times 10^{-27} \text{J}\]

\[\Delta \nu = \Delta E / h = 8 \times 10^6 \text{Hz}\]