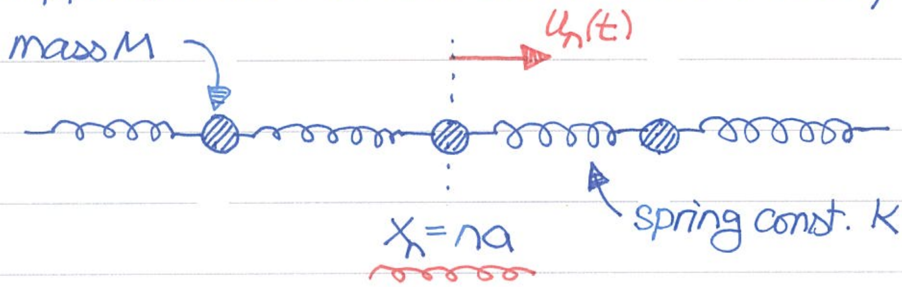
HH0101 Sound Waves

We have biological receptors for sound and light waves. Thus, it is quite important to understand these waves better ☺ Here we would like to derive the wave equation for sound waves propagating in solids and fluids.

Approximate the 1D solid as an array of coupled atoms



$u_n(t)$ is the displacement for the n^{th} atom.

$\to u(x,t) = u_n(t)$

The equilibrium position of the n^{th} atom is $x_n = na$, where a is the lattice constant. The displacement of each atom is described by the scalar field $u(x,t) = u_n(t)$.

Each atom feels to forces from its neighbors. Write down the EOM for each atom:

$$M \ddot{u}_n = -K(u_n - u_{n-1}) + K(u_{n+1} - u_n) = -K(2u_n - u_{n+1} - u_{n-1})$$

We would like to show that the above EOM is nothing but the wave equation ☺.

① Compare with previous lecture notes, reasonable.

$$\langle u \rangle = \frac{1}{2} u(x+a,t) + \frac{1}{2} u(x-a,t) = \frac{1}{2} u_{n+1} + \frac{1}{2} u_{n-1}$$

$$\to 2u_n - u_{n+1} - u_{n-1} = 2(u - \langle u \rangle) = \cancel{2} \cdot \left(-\frac{1}{\cancel{2}}\right) a^2 \frac{\partial^2 u}{\partial x^2}$$

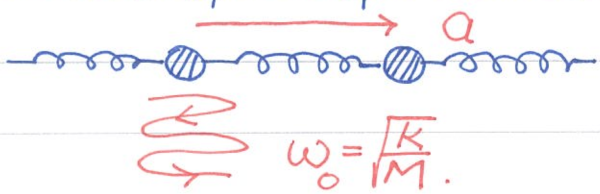
Thus, we have $\boxed{2u_n - u_{n+1} - u_{n-1} = -a^2 \frac{\partial^2 u}{\partial x^2}}$

② It is easy to see that $\ddot{u}_n = \partial^2 u/\partial t^2$.

Combine ① and ② together ooo The EOM reads

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \left(\frac{K}{M} a^2\right)\frac{\partial^2 u}{\partial x^2}} \longrightarrow \boxed{v = \sqrt{\frac{K}{M}}\, a}$$

The microscopic derivation not only shows that the dynamics is captured by the wave equation, but also the microscopic dependences of the wave speed v.

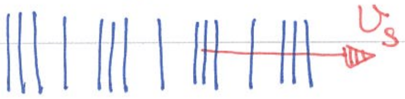


$\omega_0 = \sqrt{\frac{K}{M}}$.

$v = \sqrt{\frac{K}{M}}\, a = \omega_0 a$ the wave propagates due to micro-vibrations.

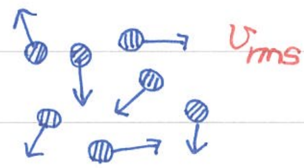
For stiff materials, the spring constant K is large → the sound speed is fast. Note that $v \sim 6000$ m/s in steel, much faster than $v \sim 330$ m/s in air.

∅ Sound waves in an ideal gas. Let us start with an interesting observation.



$v_s \sim 300$ m/s

$v_{rms} \sim 300$ m/s



The sound speed in air is roughly the same as the root-mean-square speed of the gas molecules. Is this a coincidence? Or, there is something deeper here?

Before digging into technical details, it is "healthy" to look for relevant parameters first.

relevant parameters ⇒ $P, \rho, T, m, \ldots\ldots$ What are your intuitive guesses?

Not very easy to identify the right parameters because we mix up thermodynamic and microscopic quantities.....

A rough picture for the sound wave helps...

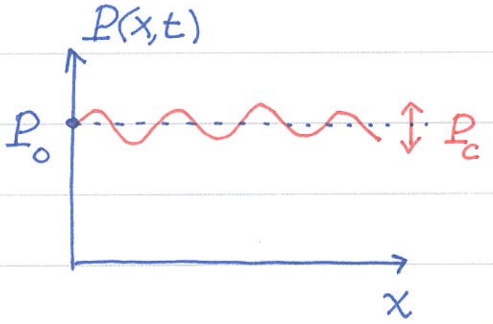
a) gas molecules move and change the density.

b) change in density $\rightarrow$ change in pressure.

c) pressure variations generate molecular motions.

The above picture suggests that we can identify P and ρ as the relevant parameters.

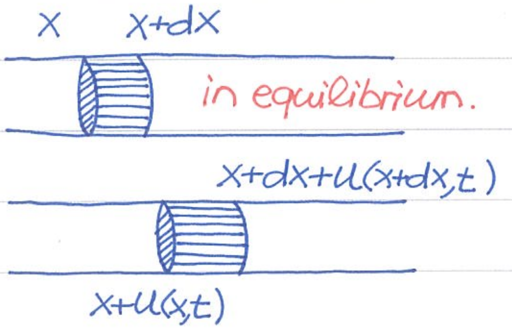
Now let's try to understand the dynamics of gas molecules,



$$\boxed{\begin{aligned} P(x,t) &= P_0 + P_c(x,t) \\ \rho(x,t) &= \rho_0 + \rho_c(x,t) \end{aligned}}$$

variations due to compression.

It is reasonable to assume that $P_c \ll P_0$ and $\rho_c \ll \rho_0$ in usual sound waves.



Assume the pressure and the density are related by some function,

$$\rightarrow \quad P = f(\rho)$$

use Taylor Expansion ...

$$\cancel{P_0} + P_c = f(\rho_0 + \rho_c) = \cancel{f(\rho_0)} + f'(\rho_0)\rho_c + \frac{1}{2!} f''(\rho_0)\rho_c^2 + \cdots$$

Keeping the lowest non-vanishing term, we obtain

$$\boxed{P_c(x,t) = K \rho_c(x,t)} \qquad K = f'(\rho_0) = \left(\frac{dP}{d\rho}\right)_0$$

Do you see the generalized Hooke's law here? $P_c \propto \rho_c$!

From mass conservation in the tiny segment:

$$\rho_0 dx = \rho\left[x + dx + u(x+dx) - (x + u(x))\right] = \rho\left[dx + \frac{\partial u}{\partial x} dx\right]$$

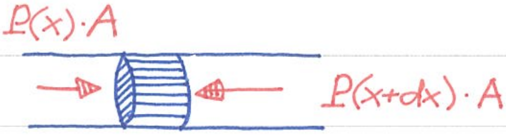
$$\rho_0 = \rho\left(1+\frac{\partial u}{\partial x}\right) = (\rho_0+\rho_c)\left(1+\frac{\partial u}{\partial x}\right)$$

Note that both $\rho_c/\rho_0 \ll 1$ and $\partial u/\partial x \ll 1$.

$$\cancel{\rho_0} = \cancel{\rho_0} + \rho_c + \rho_0\frac{\partial u}{\partial x} + \cancel{\rho_c\frac{\partial u}{\partial x}}^{\to 0}$$ negligibly small

Finally, we relate the density variation $\rho_c(x,t)$ to the spatial derivative of the scalar field,

$$\boxed{\rho_c(x,t) = -\rho_0\frac{\partial u}{\partial x}}$$



Write down the EOM for the tiny segment,

$$\left[-P(x+dx)+P(x)\right]\cancel{A} = (\rho_0\cancel{A}\cdot dx)\cdot\frac{\partial^2 u}{\partial t^2}$$

We need to simplify the LHS a bit....

$$\Rightarrow \underbrace{-P(x+dx)+P(x)}_{\text{difference in pressure.}} = -\frac{\partial P}{\partial x}dx = -\frac{\partial P}{\partial x}dx = -K\frac{\partial \rho_c}{\partial x}dx$$

$$= K\rho_0\frac{\partial^2 u}{\partial x^2}dx$$

$\Leftarrow$ 2^{nd} spatial derivative of the scalar field.

Substitute into the EOM and we obtain the wave equation,

$$K\cancel{\rho_0}\frac{\partial^2 u}{\partial x^2}\cancel{dx} = \cancel{\rho_0}\cancel{dx}\frac{\partial^2 u}{\partial t^2} \Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = K\frac{\partial^2 u}{\partial x^2}}$$ wave equation

The sound speed in the ideal gas can be expressed as

$$\boxed{v^2 = K = \left(\frac{dP}{d\rho}\right)_0}$$

Because the oscillation of the sound wave is quite fast, it can be viewed as adiabatic process (no heat in or out during oscillation).

$P = c\rho^\gamma$ where c is some const, $\gamma \equiv C_P/C_V$.

$$\frac{dP}{d\rho} = c\cdot\gamma\rho^{\gamma-1} = \frac{\gamma P}{\rho} \Rightarrow \boxed{v = \sqrt{\frac{\gamma P}{\rho}}}$$

$\Leftarrow$ Here I drop the subscript index 0.

We can also express the sound speed in terms of microscopic parameters $\Rightarrow$ see next page.

According to the ideal gas law,

$$PV = NkT \longrightarrow \boxed{P = \frac{N}{V}kT = \frac{1}{m}\rho kT}$$

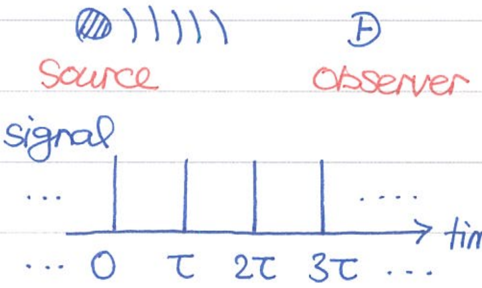
The ratio P/ρ can be replaced by kT/m,

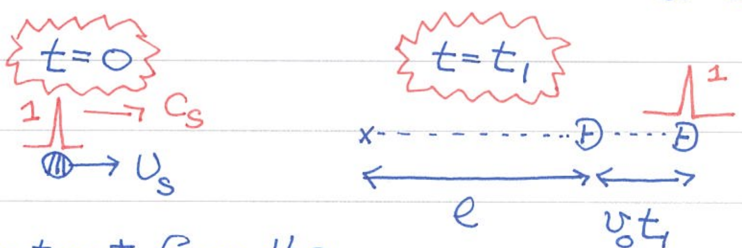
$$\boxed{v = \sqrt{\frac{\gamma kT}{m}}}$$

The sound speed is roughly the same as $v_{rms} = \sqrt{3kT/m}$.

It is rather remarkable that Newton's 2nd law reigns again and explains the propagation of sound waves. In addition, with a bit knowledge of thermodynamics, we understand why $v \sim v_{rms}$.

⊘ Doppler effect. I would like to present a simply approach to derive the Doppler effect. The usual choice is the rest frame of the medium. The velocities of the source and the observer are v_s, v_o.

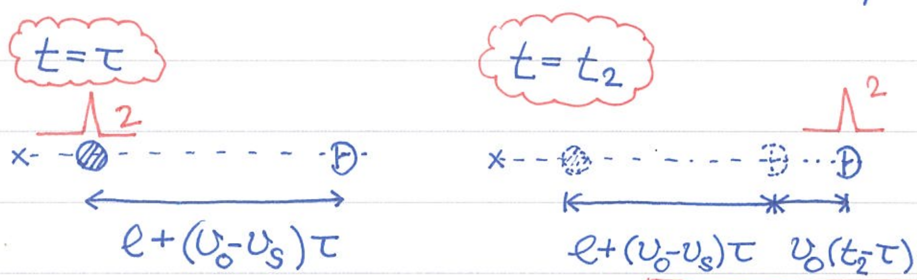




At $t=0$, the first signal is sent out from the source. At $t=t_1$, the first signal is received by the observer.

$$\ell + v_o t_1 = c_s t_1 \longrightarrow \boxed{t_1 = \frac{\ell}{c_s - v_o}}$$

Now we turn to the emission and the reception of the 2nd signal.



$$\ell + (\cancel{v_o} - v_s)\tau + v_o(t_2 - \cancel{\tau}) = c_s(t_2 - \tau)$$

$$\Rightarrow (c_s - v_o)t_2 = \ell + (c_s - v_s)\tau \qquad \boxed{t_2 = \frac{\ell + (c_s - v_s)\tau}{c_s - v_o}}$$

The time interval of the received signals is

$$\tau' = t_2 - t_1 = \frac{\ell + (c_s - v_s)\tau}{c_s - v_0} - \frac{\ell}{c_s - v_0} = \frac{c_s - v_s}{c_s - v_0}\,\tau$$

The emission interval τ and the reception interval τ' are not the same anymore. The relation between frequencies, $f = 1/\tau$, thus takes the form,

$$\boxed{f' = \frac{c_s - v_0}{c_s - v_s}\, f}$$

Doppler effect due to v_s and v_0 !

Read the textbook (or any other book) and you will know how useful and important the Doppler effect is. But, the above relation doesn't work for light. There is no such thing called "the rest frame of the medium". Only the relative velocity $v \equiv v_s - v_0$ makes sense. Writing the Lorentz transformation in matrix form

$$\boxed{\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}}$$

$$\cosh\alpha = \frac{1}{\sqrt{1-(v/c)^2}}$$

$$\sinh\alpha = \frac{1}{\sqrt{1-(v/c)^2}} \cdot \left(\frac{v}{c}\right).$$

It turns out that $(\omega, c\vec{k})$ can be viewed as a 4-vector and thus described by the same transformation,

$$\begin{pmatrix} \omega' \\ ck' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} \omega \\ ck \end{pmatrix}$$

we choose light in x direction so that $\vec{k} = (k, 0, 0)$.

$$\rightarrow \quad \omega' = \cosh\alpha\, \omega + \sinh\alpha \cdot ck = (\cosh\alpha + \sinh\alpha)\,\omega$$

The algebra is straightforward, $\cosh\alpha + \sinh\alpha = \sqrt{\frac{1+v/c}{1-v/c}}$,

$$\boxed{\omega' = \sqrt{\frac{1+v/c}{1-v/c}}\,\omega = \sqrt{\frac{c+v}{c-v}}\,\omega}$$

Doppler effect for light !

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