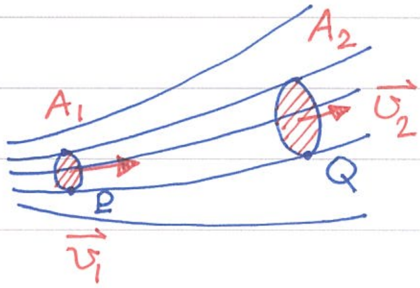
HH0098 Hagen-Poiseuille Equation.

Examples given in HH0097 can all be understood by hydrostatic when choosing appropriate reference frames. Here we would try to study dynamical behaviors in liquids.

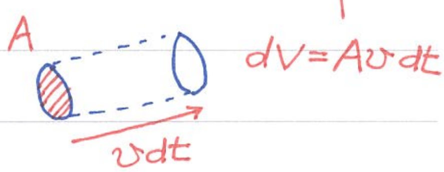


In steady flow, the velocity field $\vec{v} = \vec{v}(x,y,z)$ is independent of time. Follow a point P and it will trace out a curve, called streamline. The velocity is always tangent to the streamline as shown here. You may notice that no streamlines cross each other, Why?

∅ Continuity equation. Consider flows through the area A_1 and A_2. Because the molecules just follow the streamlines, it is easy to see that $\boxed{\dfrac{dm_1}{dt} = \dfrac{dm_2}{dt}}$, $\dfrac{dm}{dt} = \rho v A$

mass conservation

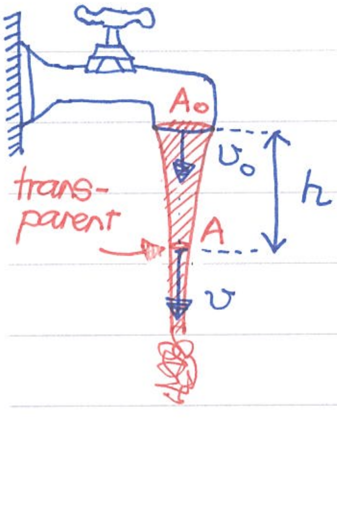
By simple kenimatic shown on the right,



$$\longrightarrow \boxed{\rho_1 v_1 A_1 = \rho_2 v_2 A_2}$$ Continuity equation.

Most fluids are nearly incompressible — $\rho(\vec{r}) \approx \rho$ is a good approximation. $\longrightarrow \boxed{v_1 A_1 = v_2 A_2} \longleftrightarrow R = \text{Const.}$

It is often convenient to introduce the volume flux $R \equiv dV/dt = Av$.



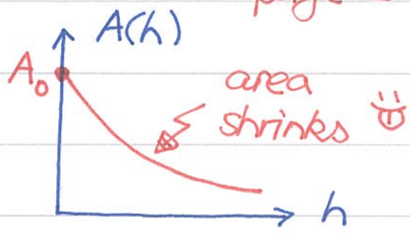
Consider the water stream from a faucet. The continuity equation gives $\boxed{v_0 A_0 = vA}$

But, from E-conservation,

$$v^2 = v_0^2 + 2gh \longrightarrow v = \sqrt{v_0^2 + 2gh}$$

Eliminate v to find the area

$$\boxed{A = A_0 \frac{v_0}{v} = A_0 \frac{v_0}{\sqrt{v_0^2 + 2gh}}}$$

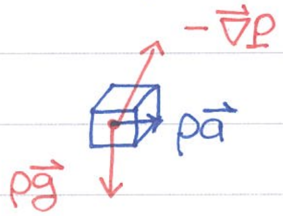


The above result is valid only when the velocity field is steady. As the flow turns turbulent, the water stream is no longer transparent due to $\vec{v} = \vec{v}(\vec{r}, t)$!

⊘ Bernoulli equation. Let us now investigate the EOM for the "tiny cube" in motion. There are two forces:

① from pressure $-\vec{\nabla}P \, dxdydz$

② from gravity $\rho\vec{g} \, dxdydz$



$dV = dxdydz$

The EOM for the tiny cube is

$$(-\vec{\nabla}P + \rho\vec{g})\, \cancel{dV} = \rho \cancel{dV} \frac{d\vec{v}}{dt}$$

$$\rightarrow \boxed{-\vec{\nabla}P + \rho\vec{g} = \rho \frac{d\vec{v}}{dt}}$$

Newton is still with us

Integrate the field equation ...

$$-\int_1^2 \vec{\nabla}P \cdot d\vec{r} + \rho \int_1^2 \vec{g} \cdot d\vec{r} = \rho \int_1^2 \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

Here we assume ρ const.

These integrals have been studied in previous lectures and the results are presented here without detail derivations,

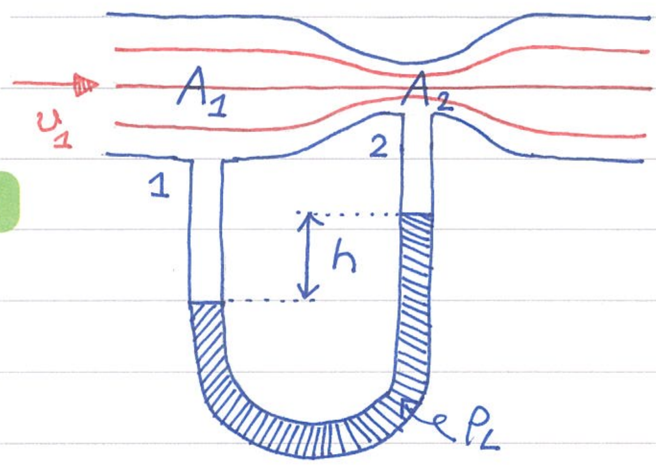
$$\rightarrow -\Delta P - \rho\Delta\Phi = \frac{1}{2}\rho\Delta v^2 \quad \text{i.e.} \quad \boxed{\Delta\left(\frac{1}{2}\rho v^2 + \rho\Phi + P\right) = 0}$$

The above result shows us that pressure P can be viewed as some sort of "potential energy density"

$$\boxed{\frac{1}{2}\rho v^2 + \rho\Phi + P = \text{const}}$$

known as Bernoulli equation. Note that it only works for incompressible fluids without viscosity.

Example. The Venturi meter. We would like to measure the speed v_1 in a pipe.



① continuity equation

$$A_1 v_1 = A_2 v_2$$

② Bernoulli equation

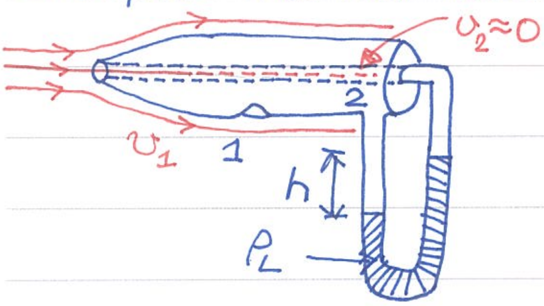
$$\frac{1}{2}\rho v_1^2 + P_1 = \frac{1}{2}\rho v_2^2 + P_2$$

The pressure difference is $P_1 - P_2 = (\rho_L - \rho) g h$, and $v_2 = (A_1/A_2) v_1$.

$$\frac{1}{2}\rho v_1^2 - \frac{1}{2}\rho\left(\frac{A_1}{A_2}\right)^2 v_1^2 + (\rho_L - \rho) g h = 0 \longrightarrow \boxed{v_1 = A_2\sqrt{\frac{2(\rho_L-\rho)gh}{\rho(A_1^2 - A_2^2)}}}$$

All parameters A_1, A_2, ρ, ρ_L are constant. One can read off the flow velocity v_1 by the height h.

Example. The Pitot tube. Another device to measure the flow velocity v_1. Only Bernoulli equation is needed here ☺

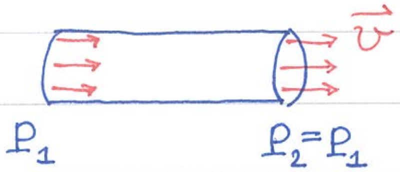


$$\frac{1}{2}\rho v_1^2 + P_1 = P_2 \quad \text{with } P_1 - P_2 = (\rho_L - \rho) g h$$

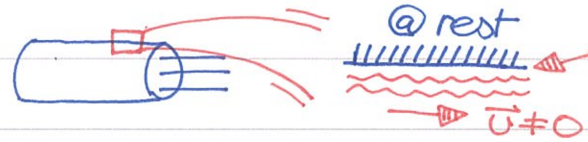
It is straightforward to find the velocity $\boxed{v_1 = \sqrt{2gh\left(\frac{\rho_L}{\rho} - 1\right)}}$

Ⅶ Viscosity in fluids.

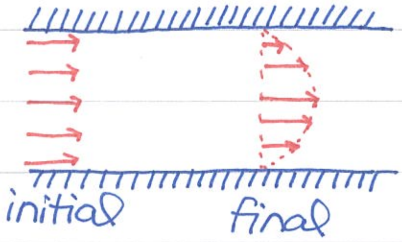
So far, we do not consider friction in fluids. It turns out to be quite important. Consider a uniform pipe with const pressure everywhere. It is easy to show that $\vec{v}$ is constant as well.



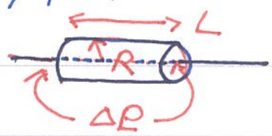
But this cannot be true....



A more reasonable guess for flows in a cylindrical pipe should be the following... The flow velocity near the boundary should be the same as the velocity of the wall (zero here). In fact, the velocity profile is

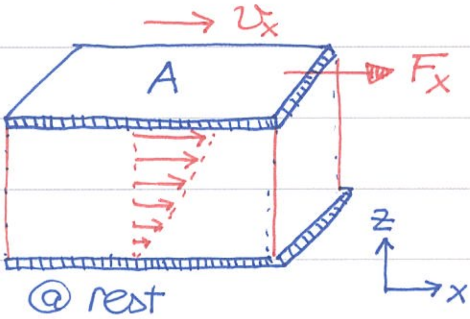


$$\boxed{v(r) = \frac{\Delta P}{4\eta L}(R^2 - r^2)}$$

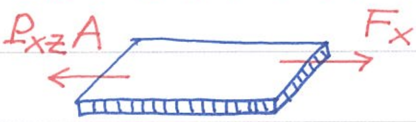


η: viscosity coefficient

In the following, we would like to learn how to derive the above velocity profile and the corresponding volume flux.



Consider the viscous flow between two slabs – the bottom one is at rest while the top one is moving at constant speed v_x. Our intuition tells us that a force F_x is needed to keep the flow steady. This force is proportional to the area A and the velocity gradient dv_x/dz.



Because the slab is moving at constant velocity, the net force is zero.

$$P_{xz}A + F_x = 0 \longrightarrow \boxed{P_{xz} = -\frac{F_x}{A} = -\eta\frac{dv_x}{dz}}$$

Notice that the area vector is $\vec{A} = (0,0,-A)$. If the pressure is a true scalar, $\vec{F} = -P\vec{A} = (0,0,PA)$ ← only $F_z \neq 0$

In a viscous liquid, the pressure is a rank-2 tensor

$$\longrightarrow \boxed{F_i = -\sum_{j=1}^{3} P_{ij}\, dA_j}$$

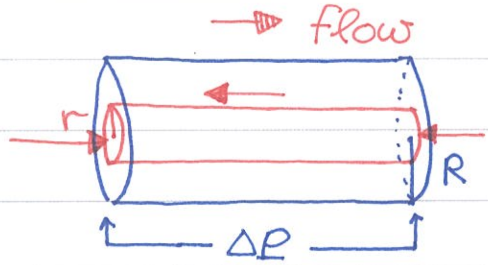
In general, the off-diagonal terms $P_{ij} \neq 0$ for $i \neq j$.

For instance, in this case, $\vec{A} = (0, 0, -A)$. Thus, the force arisen from the pressure is

$$F_i = -\sum_{j=1}^{3} P_{ij} A_j = P_{iz} A \rightarrow \vec{F} = (P_{xz}, P_{yz}, P_{zz}) A$$

Because $dv_y/dz = 0$ here, $P_{yz} = 0$. The liquid exerts the force $(P_{xz}A, 0, P_{zz}A)$ on the slab. Its direction is no longer perpendicular

Now consider the viscous flow in a cylindrical pipe of length L and radius R. Now compute the forces on an inner cylinder of radius r. Two forces show up.



① pressure difference $\Delta P \cdot \pi r^2$

② viscosity $\eta \frac{dv}{dr} \cdot 2\pi r L$ $\left(\frac{dv}{dr} < 0!\right)$

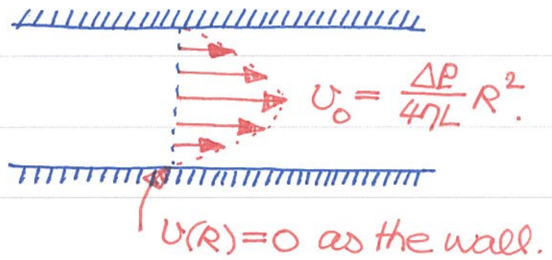
EOM: $\Delta P \cdot \pi r^2 + \eta \frac{dv}{dr} 2\pi r L = 0$

$$\rightarrow \boxed{\frac{dv}{dr} = -\frac{\Delta P}{2\eta L} r} \qquad \int_r^R \frac{dv}{dr} dr = -\frac{\Delta P}{2\eta L} \int_r^R r\, dr$$

$$\underbrace{v(R)}_{0} - v(r) = -\frac{\Delta P}{2\eta L} \frac{1}{2}(R^2 - r^2) \quad \text{finally} \quad \boxed{v = \frac{\Delta P}{4\eta L}(R^2 - r^2)}$$

The velocity profile is parabolic and its maximum at the center,

$$\boxed{v_0 = \frac{1}{4\eta}\left(\frac{\Delta P}{L}\right) R^2 \propto R^2}$$



To compute the volume flux, it's necessary to integrate over the radial direction.

$$\frac{dV}{dt} = \int v\, dA = \int_0^R v \cdot 2\pi r\, dr = \frac{\pi \Delta P}{2\eta L} \int_0^R (R^2 - r^2) r\, dr$$

The integral is basic,

$$\int_0^R (R^2-r^2)\, r\, dr = R^2\cdot\left(\tfrac{1}{2}R^2\right) - \left(\tfrac{1}{4}R^4\right) = \tfrac{1}{4}R^4$$

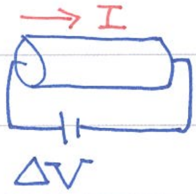
Substitute into the expression for volume flux

$$\boxed{\frac{dV}{dt} = \frac{\pi R^4}{8\eta L}\,\Delta P \propto R^4}$$

Hagen-Poiseuille equation.

Our blood circulation can be described by the Hagen-Poiseuille equation. Suppose the blood vessel is blocked with a smaller radius $R/2$. To maintain the same blood flux, the pressure need to go up $2^4 = 16$ times!

One can make interesting comparison with Ohm's law.



The electric current $I = dQ/dt$ is just the charge flux. Ohm's law states $\Delta V = I Z_e$, where Z_e is the electrical resistance. The geometric dependence of the resistance is

$$\boxed{Z_e = \rho\frac{L}{A} = \frac{\rho L}{\pi R^2} \propto \frac{1}{R^2}}$$

On the other hand, write the Hagen-Poiseuille equation as $\Delta P = \frac{dV}{dt}\cdot Z_p$,

$$\boxed{Z_p = \frac{8\eta L}{\pi R^4} \propto \frac{1}{R^4}}$$

One may interpret the difference between Z_e and Z_p by saying ΔV and ΔP drive the flux differently. Can you understand why?

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