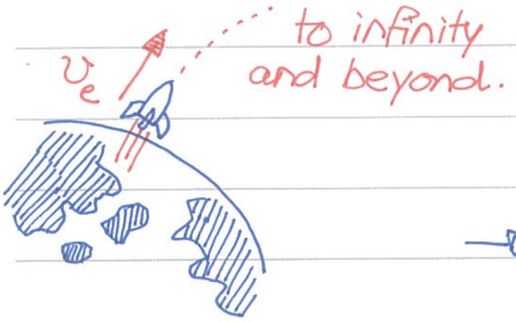
HH0095 Are Black Holes Black?

Gravity is conservative and thus $\vec{F}_g$ and U_g are related by the relations:

$$\boxed{\begin{aligned} \vec{F}_g &= \left(-\frac{\partial U_g}{\partial x}, -\frac{\partial U_g}{\partial y}, -\frac{\partial U_g}{\partial z}\right) \equiv -\vec{\nabla} U_g \\ \Delta U_g &= -\int_1^2 \vec{F}_g \cdot d\vec{r}, \quad \Delta U_g = U_g(\vec{r}_2) - U_g(\vec{r}_1) \end{aligned}}$$

The gravitational force is $\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$ and the potential energy is $U_g = -\frac{GMm}{r}$ as derived in the previous lecture. It is interesting to compute the so-called escape velocity v_e.



According to E-conservation,

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0 + 0$$

$$\rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}} \approx 10^4 \text{ m/s for Earth}}$$

Note that v_e is independent of the mass m of the object!

In Earth's atmosphere at its average temperature,

H_2: 1908 m/s
He: 1350 m/s
O_2: 477 m/s
N_2: 510 m/s
CO_2: 407 m/s

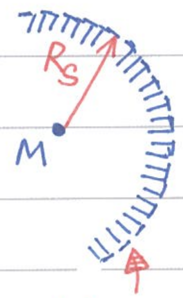
All of these molecular speeds are much smaller than the escape velocity v_e. That's why these gas molecules can be kept inside the atmosphere.

→ Q: Do you know how to estimate these?

What about trapping light? Let's take the limit $v_e \to c$.

$$c = \sqrt{\frac{2GM}{R_S}} \rightarrow \text{Schwarzschild radius } \boxed{R_S = \frac{2GM}{c^2}}$$

For a star of mass M, if its radius shrinks to $R < R_S$, it becomes a "black hole". The Schwarzschild radius for the Earth is about 1 cm



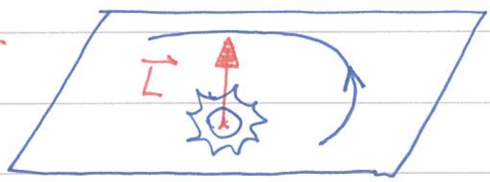
It turns out that all mass M shrinks to a point and becomes singular. Anything (including light) inside the "event horizon", $r < R_S$, cannot escape and will be attracted toward the singular point. So, black hole is actually a point — a singular point.

① Classification of orbits. Combine Newton's discoveries together: $\vec{F} = -GMm/r^2\,\hat{r}$ and $\vec{F} = m\vec{a}$

$$\boxed{-\frac{GMm}{r^2}\hat{r} = m\frac{d^2\vec{r}}{dt^2}} \longrightarrow \begin{cases} \vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{v}(t)\Delta t \\ \vec{v}(t+\Delta t) \approx \vec{v}(t) - \left(\frac{GM}{r^2}\hat{r}\right)\Delta t \end{cases}$$

So, from given $\vec{r}(t), \vec{v}(t)$, we can find $\vec{r}(t+\Delta t), \vec{v}(t+\Delta t)$. This can be done easily by numerical methods. Repeating the iterations, we can trace out the trajectory.

First of all, because the torque vanishes $\vec{\tau} = \vec{r} \times \vec{F}_g = 0$, the angular momentum is constant.

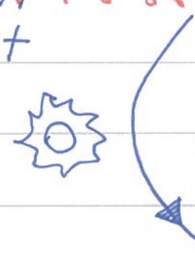


Thus, $\vec{L}$ determine the orbital plane and also the evolving angular speed. Secondly, the energy plays a crucial role in determining orbital types.

E<0

ellipse.

$E<0$, the orbit is an ellipse and the Sun is located at the focus. On the other hand, for $E>0$, the orbit is a hyperbola with the Sun at the focus. The orbit is OPEN in this case.



E>0

hyperbola

Q: What about $E=0$ case?

⑦ Gravitational field and potential.

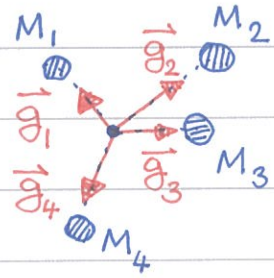
The gravitational force on an object is proportional to its mass m. Thus, we can define the gravitational field $\vec{g}$

$$\vec{F} = m\vec{g} \longrightarrow \boxed{\vec{g} = -\frac{GM}{r^2}\hat{r}}$$

Similarly, we can define the gravitational potential.

$$U = m\Phi \longrightarrow \boxed{\Phi = -\frac{GM}{r}}$$ with the choice $\phi(r\to\infty) = 0$

When more than one gravitational sources are present, the field is obtained by the vector sum,



$$\boxed{\vec{g} = \vec{g}_1 + \vec{g}_2 + \vec{g}_3 + \cdots = \sum_i \vec{g}_i}$$

BUT! The vector sum may be tough to compute.
The gravitational potential comes to rescue ت

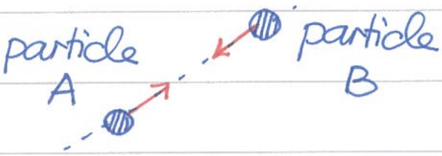
$$\boxed{\Phi = \sum_i \Phi_i = \left(-\frac{GM_1}{r_1}\right) + \left(-\frac{GM_2}{r_2}\right) + \cdots}$$

It's usually much easier to add scalars.

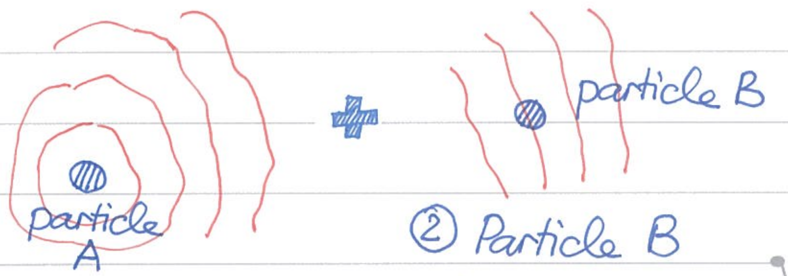
Once we know the potential, it is easy to compute the field.

$$\boxed{\vec{g} = -\vec{\nabla}\Phi = \left(-\frac{\partial\Phi}{\partial x}, -\frac{\partial\Phi}{\partial y}, -\frac{\partial\Phi}{\partial z}\right)}$$ ← same as $\vec{F}_g = -\vec{\nabla}U_g$.

The concept of "field" turns out to be important.



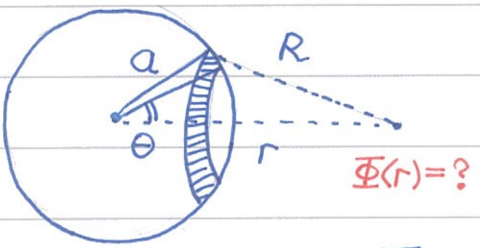
Newton's view.
instantaneous long-distance interaction



① Particle A builds up the field.

② Particle B interacts with the local field.

① Shell theorem. Let us compute the gravitational potential of a spherical shell.



$$\boxed{\begin{aligned}\Phi &= \sum_i -\frac{GM_i}{R_i} \\ &= \int -\frac{G}{R}\,dM\end{aligned}}$$

The surface density $\sigma = M/4\pi a^2$.

The infinitesimal area element is $dA = \underbrace{(2\pi a \sin\theta)}_{\text{length}} \times \underbrace{(a\,d\theta)}_{\text{width.}}$

Thus, the mass of the ring is

$$dM = \sigma\, dA = \frac{M}{4\pi a^2}\cdot 2\pi a^2 \sin\theta\, d\theta = \frac{1}{2} M \sin\theta\, d\theta$$

Substitute into the integral presentation for the potential,

$$\boxed{\Phi = -\int \frac{G\,dM}{R} = -\frac{1}{2} GM \int \frac{\sin\theta}{R}\, d\theta}$$

Change variable $\theta \to R$ ☺.

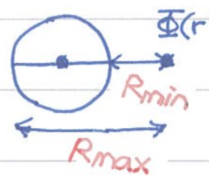
Note that $|\vec{R}|^2 = |\vec{r}-\vec{a}|^2 = r^2 + a^2 - 2\vec{r}\cdot\vec{a}$ from the figure.

$$\rightarrow \boxed{R^2 = a^2 + r^2 - 2ar\cos\theta} \rightarrow 2R\,dR = 2ar\sin\theta\, d\theta$$

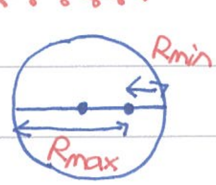
integrand $\dfrac{\sin\theta}{R}\, d\theta = \dfrac{dR}{ar}$ $\rightarrow$ $\boxed{\Phi = -\dfrac{GM}{2ar}\displaystyle\int dR}$ simple!

① $r > a$, $R_{min} = r - a$, $R_{max} = r + a$.

$$\Phi = -\frac{GM}{2ar}\int_{r-a}^{r+a} dR = -\frac{GM}{\cancel{2a}r}\,\cancel{2a} = -\frac{GM}{r}$$



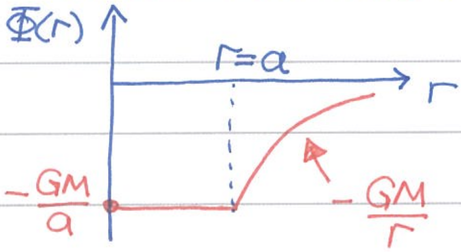
② $r < a$, $R_{min} = a - r$, $R_{max} = a + r$



$$\Phi = -\frac{GM}{2ar}\int_{a-r}^{a+r} dR = -\frac{GM}{\cancel{2}a\cancel{r}}\cdot\cancel{2r} = -\frac{GM}{a}$$

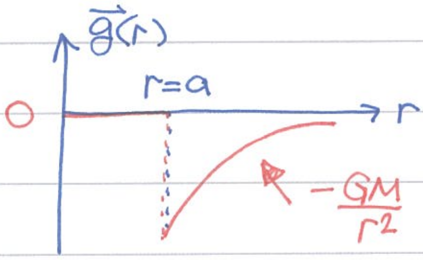
The gravitational potential outside the shell is the same as that of a point mass at the center. And, inside the shell, the potential is constant!

Collect the results and make the plot for $\Phi(r)$.



$$\boxed{\Phi(r) = \begin{cases} -\dfrac{GM}{r}, & r>a \\ -\dfrac{GM}{a}, & r\leq a \end{cases}}$$

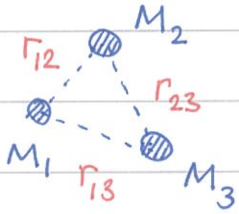
The gravitational field can be computed by $\vec{g} = -\vec{\nabla}\Phi$. Note that we have compute the gradient in previous lecture: $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r}$ ◄— important



$$\boxed{\vec{g}(r) = \begin{cases} -\dfrac{GM}{r^2}\hat{r}, & r>a \\ 0 & r<a \end{cases}}$$

It is quite interesting that $\vec{g}=0$ everywhere inside the shell!

⊘ Gravitational potential energy. It is straightforward to compute the potential energy in this case ($N=3$ here).



$$U_g = -\frac{GM_1M_2}{r_{12}} - \frac{GM_2M_3}{r_{23}} - \frac{GM_3M_1}{r_{13}}$$

$$= \frac{1}{2}\left(M_1\Phi_1 + M_2\Phi_2 + M_3\Phi_3\right)$$

→ remove the double counting

Generalize the reasoning to arbitrary N-particle system,

$$\boxed{U_g = \frac{1}{2}\sum_i M_i\Phi_i = \frac{1}{2}\int \Phi\, dM}$$

◄— Apply it to the spherical shell of mass M.

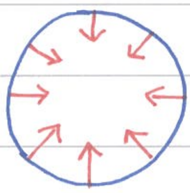
$$U_g = \frac{1}{2}\int \Phi\, dM = -\frac{GM}{2a}\int dM \longrightarrow \boxed{U_g = -\frac{GM^2}{2a}}$$

This can also be understood by building up the mass bit by bit.

$$dU_g = -\frac{Gm}{a}dm \longrightarrow U_g = \int_0^M -\frac{Gm}{a}dm = -\frac{GM^2}{2a}$$

Without other forces, the shell tends to collapse. Let us try to compute the negative pressure due to gravitational attraction.

$$\rightarrow \quad dU_g = -F_g\,da = -P\,4\pi a^2\,da$$

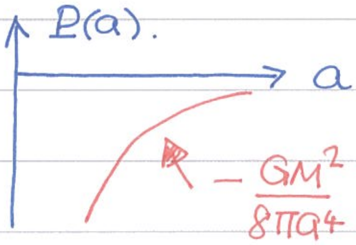


negative pressure!

We already know $U_g = -GM^2/2a$.

$$P = -\frac{1}{4\pi a^2}\frac{dU_g}{da} = -\frac{1}{4\pi a^2} \times \frac{GM^2}{2a^2}$$

$$\rightarrow \quad \boxed{P = -\frac{GM^2}{8\pi a^4} \propto \frac{1}{a^4}}$$



Without other supporting forces, all mass will collapse into one singular point. If this happens, we have a black hole. Luckily, the gravitational collapse does not occur very often.

清大東院
2013.1119