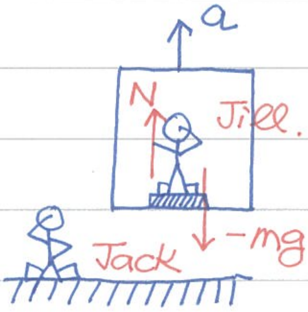
HH0094 Coriolis Effect

In a non-inertial frame, Newtonian mechanics needs corrections. Fictitious force must be included to make the EOM right.

Suppose Jill stands on a scale and both moving up with an acceleration $a \neq 0$. Jack stays on the ground and tries to understand Jill's motion by writing down the EOM:



Jack's view $\quad N - mg = m\dfrac{dv}{dt}$ He observes $\dfrac{dv}{dt} = a$.

$\rightarrow$ $N = m(g+a)$ The scale reading N is larger than mg because it needs to accelerate Jill's system.

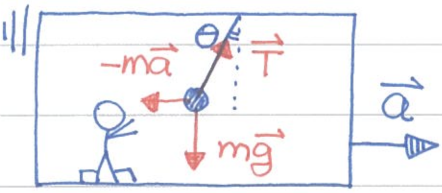
Jill's view $\quad$ She observes $\dfrac{dv}{dt} = 0$, but $N - mg \overset{?}{=} m\dfrac{dv}{dt}$

This would be in contradiction's with Jack's conclusion. Because Jill is in a non-inertial frame with acceleration a, one needs to add a fictitious force $-ma$ to EOM.

$\rightarrow$ $N - mg + \mathcal{F} = m\dfrac{dv}{dt} = 0$ $\quad$ where $\boxed{\mathcal{F} = -ma}$

Jill will obtain $N = mg - \mathcal{F} = m(g+a)$ – consistent with Jack's conclusion ☺

⊘ Fictitious force v.s. gravity. Now Jill is in a moving train with acceleration $\vec{a}$. She observes that a static pendulum is tilted by an angle θ.



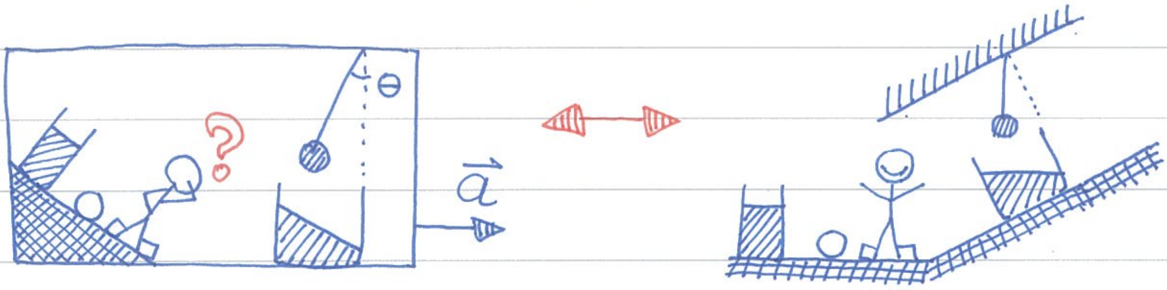
$$\boxed{\vec{T} + m\vec{g} + \vec{\mathcal{F}} = m\frac{d\vec{v}}{dt} = 0}$$

Again, the fictitious force is $\vec{F_i} = -m\vec{a}$

$\rightarrow$ $\vec{T} + m\vec{g} - m\vec{a} = 0$, $\quad \vec{T} = m(\vec{g} - \vec{a})$

From the force diagram, it is easy to find the tilting angle $\boxed{\tan\theta = \dfrac{a}{g}}$ $\quad 0 \le \theta < \dfrac{\pi}{2}$

Jill now performs more experiments as shown below.



fictitious force $\vec{F_i} = -m\vec{a}$ $\qquad$ effective gravity $\vec{g}_{eff} = \vec{g} - \vec{a}$

It seems that Jill can think in two different ways:

(1) Introduce $\vec{F_i} = -m\vec{a}$ because it's not an inertial frame.

(2) Still an inertial frame, but gravity is modified $\vec{g} - \vec{a}$.

Einstein tells us that both pictures are equivalent and you cannot tell the difference ☺

① Typhoon, inertial circle and drain vortex. The earth

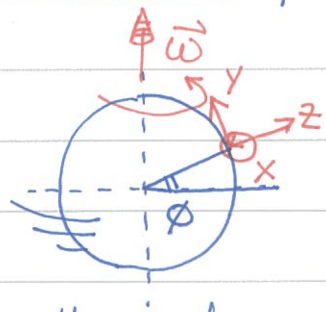
is rotating with angular velocity $\vec{\omega}$. Thus, for a person on the ground, he is not in an inertial frame. After some derivations it can be shown that the fictitious force is

$\rightarrow$ $$\boxed{\vec{F_i} = -2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

Coriolis force $\qquad$ centrifugal force.

(like a magnetic field)

Choose the ϕ-dependent local coordinates,



$$\vec{\omega} = \begin{pmatrix} 0 \\ \cos\phi \\ \sin\phi \end{pmatrix} \omega$$ ← angular velocity, a vector!

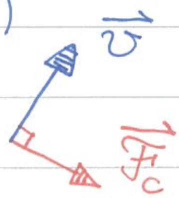
The Coriolis force $\vec{F}_c$ can be computed by the outer product, $\vec{F}_c = -2m\vec{\omega}\times\vec{v} = 2m\vec{v}\times\vec{\omega}$

$$\vec{F}_c = m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & \cos\phi & \sin\phi \end{vmatrix} \times 2\omega = 2m\omega \begin{pmatrix} v_y\sin\phi - v_z\cos\phi \\ -v_x\sin\phi \\ v_x\cos\phi \end{pmatrix}$$

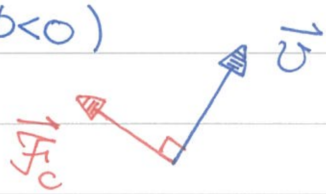
Most of the time, we are interested in the motion on the x-y plane. Thus, we set $v_z = 0$ and project the vectors onto the x-y plane.

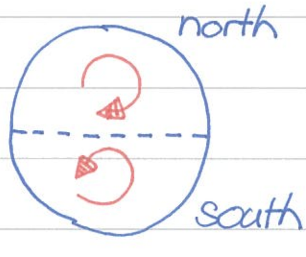
$$\boxed{\vec{F}_c = 2m\omega\sin\phi \begin{pmatrix} v_y \\ -v_x \end{pmatrix}}$$ $\vec{F}_c \cdot \vec{v} = 0$!

north hemisphere ($\phi > 0$)



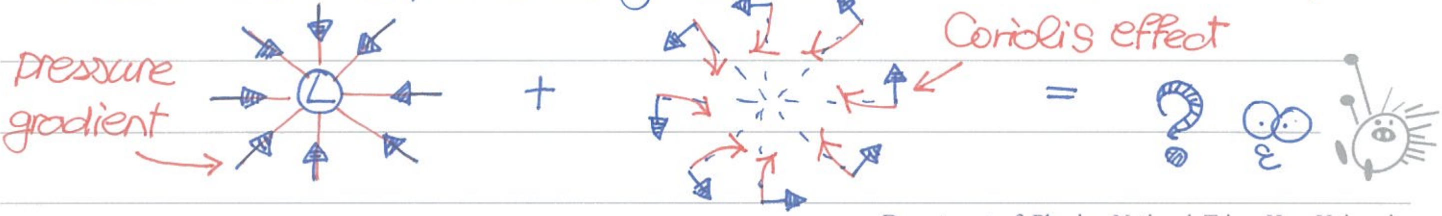
South hemisphere ($\phi < 0$)



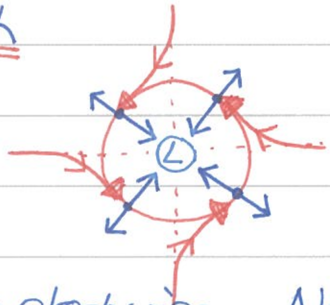


Without any other force except $\vec{F}_c$, the flow is clockwise in northern hemisphere, while it becomes counter-clockwise in southern hemisphere. These flows are "inertial circles".

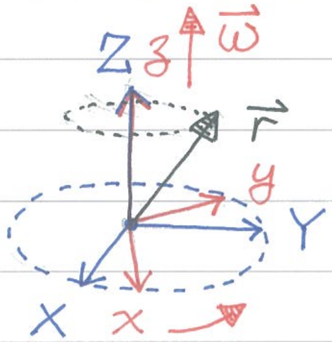
What about typhoons? One needs to consider both Coriolis effect and the pressure gradient. In Taiwan ($\phi > 0$),



Combine both influences together, the force due to the pressure gradient points inward while the Coriolis force points outward. The flows generate a typhoon spinning counter-clockwise. Note that the flow direction is opposite to that of the inertial circle. Finally, a brief comment on the direction of drain vortex in a bathtub. Its direction is usually not related to Coriolis effect.



Ⓓ Derivation of Coriolis effect. Suppose Jack is at rest and Jill is spinning with angular velocity $\vec{\omega}$. The observed velocities are different.

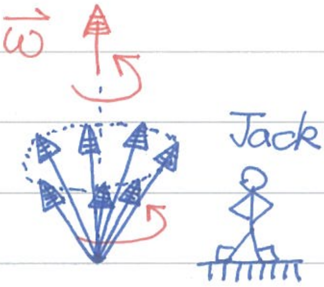


$$\boxed{\begin{aligned}\vec{V} &\equiv \left(\frac{d\vec{r}}{dt}\right)_{Jack} \quad \leftarrow \text{inertial frame}\\ \vec{v} &\equiv \left(\frac{d\vec{r}}{dt}\right)_{Jill} \quad \leftarrow \text{rotating frame.}\end{aligned}}$$

When $\vec{\omega}=0$, $\vec{V}=\vec{v}$. Thus, we expect the velocities are related by

$$\left(\frac{d\vec{r}}{dt}\right)_{Jack} = \left(\frac{d\vec{r}}{dt}\right)_{Jill} + \text{correction}$$

It is easy to find the "correction" by considering a constant vector $\vec{r}$ in Jill's frame. In Jack's view, the vector $\vec{r}$ is rotating as shown on the left.



$$\left(\frac{d\vec{r}}{dt}\right)_{Jack} = \vec{\omega}\times\vec{r}$$

The correction due to Jill's rotation is simply $\vec{\omega}\times\vec{r}$!

Thus, we arrive at the important relation:

$$\rightarrow \boxed{\left(\frac{d\vec{r}}{dt}\right)_{Jack} = \left(\frac{d\vec{r}}{dt}\right)_{Jill} + \vec{\omega}\times\vec{r}}$$

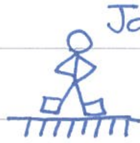
Or, equivalently, it can be expressed as

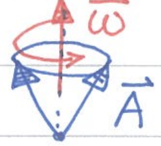
$$\boxed{\vec{V} = \vec{v} + \vec{\omega}\times\vec{r}}$$

$\vec{v}$-correction $= \vec{\omega}\times\vec{r}$

If one rethinks the logic carefully, it shall be clear that similar relation holds good for arbitrary vector $\vec{A}$.

$$\boxed{\left(\frac{d\vec{A}}{dt}\right)_{Jack} = \left(\frac{d\vec{A}}{dt}\right)_{Jill} + \vec{\omega}\times\vec{A}}$$





Apply the above relation to derive the relation between accelerations observed by Jack and Jill.

$$\left(\frac{d\vec{V}}{dt}\right)_{Jack} = \left(\frac{d\vec{V}}{dt}\right)_{Jill} + \vec{\omega}\times\vec{V} \leftarrow \vec{V} = \vec{v} + \vec{\omega}\times\vec{r}$$

$$= \left(\frac{d\vec{v}}{dt}\right)_{Jill} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{Jill} + \vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Because $(d\vec{r}/dt)_{Jill} = \vec{v}$, the relation between accelerations is

$$\boxed{\left(\frac{d\vec{V}}{dt}\right)_{Jack} = \left(\frac{d\vec{v}}{dt}\right)_{Jill} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

The first correction term gives Coriolis force and the second correction term delivers centrifugal force.

Jack's EOM: $\vec{F} = m\left(\frac{d\vec{V}}{dt}\right)_{Jack}$.

Jill's EOM: $\vec{F} + \vec{\mathcal{F}}_I = m\left(\frac{d\vec{v}}{dt}\right)_{Jill}$

$$\boxed{\vec{\mathcal{F}}_I = -2m\vec{\omega}\times\vec{v} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

fictitious force in rotating frame

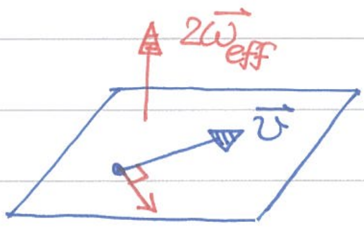
Centrifugal force $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is rather easy to be understood. $\rightarrow$ Focus on $\vec{F}_c = -2m\vec{\omega}\times\vec{v}$

Reverse the outer product to absorb the minus sign,

$$\boxed{\vec{F}_c = m\,\vec{v}\times(2\vec{\omega})} \longleftrightarrow \vec{F}_m = q\,\vec{v}\times\vec{B}$$

Coriolis force is very similar to the Lorentz force for a charged particle in magnetic field.

As calculated before, projected Coriolis force on the x-y plane is $\vec{F}_c = 2m\omega\sin\phi\,(v_y, -v_x)$



northern hemisphere

One can view this with an effective angular velocity $\boxed{2\vec{\omega}_{eff} = 2\omega\sin\phi\,\hat{z}}$

At the equator ($\phi=0$), there is no Coriolis effect on the horizontal x-y plane $\rightarrow$ no typhoon there ☺

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